Instructor Supplement for Logic: The Basics

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PREFACE

This supplement provides sample answers to most of the unanswered exercises in *Logic: The Basics* (Routledge, 2010).[1] We skip a few exercises, those for which the answers are plainly given in the main body of the book. Chapters 1–12 of this supplement correspond to Chapters 1–12, respectively, of *Logic: The Basics*.

In addition to answers to exercises, we briefly present the idea of so-called *tableau systems* (see Ch. 13), and in turn present adequate tableau systems for the various logics discussed in *Logic: The Basics*. These tableau systems can be – and their 'reason for being' is that they be – *useful* in figuring out what follows from what according to a given logical theory (i.e., figuring out what arguments are valid according to a given logical theory).

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Suggestions for other supplemental material are welcome.

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[1] http://www.routledge.com/books/details/9780415774994/.

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CONTENTS

PART I

BACKGROUND IDEAS

1

Consequences

We provide examples of answers to problems in exercise 7. (Later chapters provide answers to more of the given chapter's problems.) Georg Brun [GB] and Tim Rugalsky [TR] provided useful notes on these answers that we herein use.

One important note, in general, is worth emphasizing from the start, a point that affects many exercises and runs throughout the book: the sometimes not-easy-to-see-quickly distinction between *being untrue in a case* and *being false in a case*. The general recipe for *validity* that is used throughout the book is given in terms of the *former* notion, not the latter. So, for example, nothing in the recipe for validity requires that a sentence be *false-in-situation-c* simply because it is *untrue-in-c*. Whether this 'distinction' ought to collapse is a hard, substantial issue in philosophical logic: it concerns the logical behavior of negation (and one's theory of falsity, etc.). These issues are too hard for this book, but they are some of the issues that the book is intended to spark as students get a feel for some standard philosophical logics and, importantly, get a taste of logical theorizing on different fronts.[1]

With respect to exercise 7, it is important to have students be clear on the task—for example, specifying, as clearly as possible, what notion of *possibility* they assume in their answer(s). That possibilities are 'situations' of which we can coherently conceive is one natural thought, but there are many—many—other notions one might have. What is important at this stage is that students be clear about what notion they're assuming. (We highlight this in 7b.)

Exercises

1. What is an argument?
2. What is a valid argument?
3. What is a sound argument?
4. What is the general 'recipe' for defining logical consequence (or validity)? What are the two key ingredients that one must specify in defining a consequence relation?

[1] Perhaps, as Georg Brun noted in correspondence, this important point concerning *falsity in a case* and *untruth in a case* should've been emphasized more in the main body of the text in Chapter 1. We—well, Beall—decided against this, thinking that too many distinctions from the start would be too much. The distinction is gently introduced in an exercise in Chapter 2, and then explicitly discussed in Chapters 4–6, where it plays a crucial role. Whether Beall made the right pedagogical decision is an open question. We flag the point here, should your students require a discussion earlier than Chapter 2 or Chapter 4.

5. Consider the 'necessary consequence' relation, which takes cases to be possibilities. Assume, as is reasonable (!), that our actual world is possible—that is, that whatever is true (actually true) is possibly true. Question: on this account of logical consequence, are there any sound arguments that have false conclusions? If so, why? If not, why not?

   Answer. This is sufficiently answered in Sample Answers.

6. As noted in the chapter, 'if and only if' (which is often abbreviated as 'iff') expresses two conditionals: 'A iff B' expresses both of the following conditionals.[2]
   - If A, then B.
   - If B, then A.

   For our purposes, a biconditional 'A iff B' is true so long as A and B are either both true or both false (and such biconditionals are false otherwise). With this in mind, consider the necessary consequence relation. Is the following argument valid (where, here, validity is necessary consequence)? If it is valid—if its conclusion is a necessary consequence of the premises—explain why it is valid. If not, explain why not.
   (a) Max is happy if and only if Agnes is sleeping.
   (b) Agnes is sleeping.
   (c) Therefore, Max is happy.

   What about the following argument?
   (d) Max is happy if and only if Agnes is sleeping.
   (e) Agnes is not sleeping.
   (f) Therefore, Max is not happy.

   Answer. This is sufficiently answered in Sample Answers.

7. Using the 'necessary consequence' account of validity, specify which of the following arguments are valid or invalid. Justify your answer.
   (a) Argument 1.
      i. If Agnes arrived at work on time, then her car worked properly.
      ii. If Agnes's car worked properly, then the car's ignition was not broken.
      iii. The car's ignition was not broken.
      iv. Therefore, Agnes arrived at work on time.

      Answer [GB]. On our necessary consequence account of validity, this is not a valid argument. To show this, we have to provide a counterexample; that is, a situation we can coherently conceive of, in which (iv) is not true, but (i)–(iii) all are. Such a situation may be characterized

[2]Strictly speaking, what is expressed is the 'conjunction' of the two conditionals, but we leave the notion of *conjunctions* for the next chapter.

as follows: (i) we assume that there is no other way for Agnes to arrive at work on time, except by using her properly working car. (ii) we also assume that Agnes's car has a 'normal' engine that needs a functional ignition to work properly (it is, for example, no electromobile or diesel); so (ii) is true as well. But even if we further assume that (iii) is true and her car's ignition is not broken, there are still other possible causes for her not to arrive at work on time (her car has run of petrol, the road is blocked, she does not feel like going to work). In these cases, (iv) is not true.

(b) Argument 2.

i. Either the sun will rise tomorrow or it will explode tomorrow.
ii. The sun will not explode tomorrow.
iii. Therefore, the sun will rise tomorrow.

<u>Answer</u> *[TR].* There are different answers that one might give, depending on one's notion of *possibility*. Here are some. [NB: one could also assume an *exclusive* use of 'or', which would change things. Throughout these supplemental notes and the book itself, we expect students to use an inclusive reading of 'or'.]

i. 7b is valid. Proof: consider as possible circumstances all those in which exactly one of the following is true: the sun will explode tomorrow; the sun will not explode tomorrow. In this interpretation, when one is true the other is false, and vice versa. Now consider any possibility in which (i) and (ii) are true. Since (ii) is true, it must be false that the sun will explode tomorrow. Then, for (i) to be true, it must be the case that the sun will rise tomorrow. Therefore, under this interpretation of what is possible, the argument is valid.
ii. 7b is invalid. Counterexample: consider the possible case in which the sun explodes at the precise boundary in time between today and tomorrow. Interpret that moment as both tomorrow and today (which is not tomorrow). Thus the sun will explode tomorrow, making premise (i) true. Also, the sun explodes today, which is not tomorrow, making premise (ii) true. Finally, since the sun has exploded long before sunrise, it will not rise tomorrow, making the conclusion false. Thus, under this interpretation of what is possible, the argument is invalid.
iii. 7b is vacuously valid. Explanation: under several interpretations of what is 'possible', it is impossible for both premises to be true. (In the following examples, it is also impossible for the conclusion to be true, but this happens to be irrelevant.)
    - e.g. It is not possible for the sun literally to 'rise.' We see it on the horizon due to the earth's rotation.

- e.g. At the earth's poles, there are seasons in which the sun does not rise at all (neither does it set). If I am located at a pole, I might reasonably define possible cases as those which I experience.

In both examples, 'The sun will rise tomorrow' cannot be true under any possible circumstance. Thus, for any possible case in which premise (ii) is true, premise (i) is false. That is, there is no possible case in which all the premises are all true. Thus (vacuously), there is no possible case in which the premises are all true but the conclusion is false. The argument satisfies the definition of validity—but only in a technical, vacuous sense.

(c) Argument 3.

i. If Max wins the lottery, then Max will be a millionaire.
ii. Max will not win the lottery.
iii. Therefore, Max will not be a millionaire.

Answer. [TR] The argument is invalid. Counterexample: Max will not win the lottery, but Max is a millionaire due to inheritance.

(d) Argument 4.

i. If Beetle is an extraterrestrial, then Beetle is not from earth.
ii. Beetle is an extraterrestrial.
iii. Therefore, Beetle is not from earth.

Answer [TR]. The argument is valid. Consider any possible circumstance in which premises (i) and (ii) are both true. Then it is true that Beetle is an extraterrestrial (from ii). Since Beetle is an extraterrestrial, then Beetle is not from earth (from i). Thus Beetle is not from earth, and the conclusion is true.

Sample answers

Answer 5. On the necessary-consequence sense of 'validity' (the sense in question), an argument is *valid* iff every possibility (e.g., possible circumstance) in which the premises are all true is one in which the conclusion is true. Hence, if the actual world—the 'real' world, the way things really are—counts as a possibility, then it itself cannot be a case in which the premises of a valid argument are true but the conclusion false. But, then, any *sound* argument—that is, a valid argument whose premises are all (actually) true—is one in which the conclusion is true, and so not false.[3]

Answer 6. The argument from (6a) and (6b) to (6c) is valid in the necessary-consequence approach to validity: it is not possible for both of (6a) and (6b) to

[3] This last step—from *true* to *not false*—is something that some logical theories reject, but these theories are left for later chapters.

be true without (6c) being true. After all, recall that (6a) expresses not only that *if Max is happy then Agnes is sleeping*; it also expresses that *if Agnes is sleeping then Max is happy.* Now, consider any possibility (and possible circumstance) in which both (6a) and (6b) are true, that is, a possible circumstance in which not only Agnes is sleeping, but *if Agnes is sleeping (in that circumstance), then Max is happy (in that circumstance).* Well, then, no matter what possible circumstance we choose, it'll be one in which Max is happy if it's one in which both (6a) and (6b) are true. (Of course, there are, presumably, many possibilities in which neither (6a) nor (6b) are true, but this does not affect the necessary-consequence sense in which the given argument is valid. Why?)

2

Language, Form, and Logical Theories

We provide examples of some answers to some problems. (Later chapters provide answers to more of the given chapter's problems.) As with Chapter 1, Georg Brun [GB] and Tim Rugalsky [TR] provided useful notes on these answers that we herein use.

Exercises

1. What is a sentential connective? What is a unary connective? What is a binary connective? (What is the degree or arity of a sentential connective?)
2. Relying on the informal idea of 'possible circumstance' for our 'cases', and using the 'truth condition' in §2.4 for conjunction, say whether the following argument form is valid: $A \land B \therefore B$. Justify your answer by invoking the general definition of 'validity' (or logical consequence) and the given truth condition.
3. In §2.4 we gave a natural truth condition for conjunction. Give what you'd take to be a natural 'truth condition' (strictly, *truth-in-a-case* condition) for disjunction. Do the same for negation. (You'll need these conditions in some of what follows.)
4. Consider the argument from premises (6) and (9) to conclusion (4). Using the symbolism introduced above, give its argument form. Taking 'cases' to be 'possible circumstances', and using the truth conditions that you provided for disjunction and negation (and, if need be, the condition in §2.4 for conjunction), is the given form valid? Justify your answer.

   <u>Answer</u> *[GB & TR].* The argument is:

   6. Max likes beans or Agnes likes beans (or both).
   9. It is not true that Max likes beans.
   4. Therefore, Agnes likes beans.

   The logical form is $A \lor B, \neg A \therefore B$. This argument form is valid if we assume the following star principle:[1]

   * If a sentence is false-in-c it is not true-in-c.

   Proof. We show that there cannot be a counterexample; that is, no possible circumstance c in which $A \lor B$ and $\neg A$ are both true, but B is not

[1]This principle is a reasonable but big assumption, one challenged by logical theories explored in later chapters. Note that if your answer to the question concerning negation's truth conditions involved something along the following star-principle lines, then your account will also be challenged in later chapters. For now, things are left sufficiently open to allow for divergent views. The important thing, as in this answer, is to be clear about your assumptions.

true. According to the truth conditions for disjunctions, if c is a possible circumstance in which $A \vee B$ is true, then we have one of the following three:

i. A is true-in-c, but B is not true-in-c
ii. B is true-in-c, but A is not true-in-c
iii A is true-in-c and B is true-in-c.

If c is a possible circumstance such that $\neg A$ is true-in-c as well, then, according to the truth conditions for negations, A is false-in-c and according to (*) A is not true-in-c. This rules out (i) and (iii) as counterexamples. Hence, if c is a counterexample, (ii) must hold; that is, B is true-in-c. But then c is not a possible circumstance in which the conclusion B is not true and consequently, c is not a counterexample. If we reject (*), $\neg A$ does not place any restriction on counterexamples with respect to (i)–(iii). So we can construct a counterexample c with B not true-in-c, A true-in-c, and A false-in-c. $A \vee B$ is then true-in-c in virtue of (i), $\neg A$ is true-in-c because A is false-in-c and B is not true-in-c.

5. Consider the argument form $\neg A \vee B, A \therefore B$. Taking 'cases' to be 'possible circumstances', and using the truth conditions that you provided for disjunction and negation, is the given form valid? Justify your answer. (Your answer may turn, in part, on your philosophy of 'possible circumstances'!)
<u>*Answer*</u> *[GB] and [TR].* One answer to exercise 3 gives the truth conditions for negations and disjunctions in such a way that a counterexample c must be a possible circumstance with the following characteristics:
   (a) Given disjunction conditions, one of the following three holds:
      i. $\neg A$ is true-in-c but B is not true-in-c;
      ii. $\neg A$ is not true-in-c but B is true-in-c;
      iii. $\neg A$ and B are both true-in-c.

      According to the truth conditions for negations, these amount to:
      i. A is false-in-c and B is not true-in-c;
      ii. A is not false-in-c and B is true-in-c;
      iii. A is false-in-c and B is true-in-c.
   (b) A is true-in-c.
   (c) it is not the case that B is true-in-c.

   To prove validity, we need the following 'plus' assumption:
   + If a sentence is true-in-c it is not false-in-c.

   According to (+), (b) rules out (i) and (iii); Furthermore, (c) rules out (ii) and (iii). Hence there cannot be a counterexample meeting all three conditions (a)–(c) and consequently, the given argument form is valid. If we reject (+), (b) no longer rules out (i) and (iii). We can then construct a counterexample c as follows: A is true-in-c, A is false-in-c, and B is not true-in-c.

6. Let us say that a *sentence* is *logically true* if and only if there is no case in which it is not true. Using the truth conditions that you gave for disjunction and negation, say whether the disjunction of (2) and (8) is logically true. Justify your answer. (Also, what is the logical form of the given sentence? Is it true that, given your truth conditions, *every* sentence of that form is logically true?)

   Answer. *[TR]* The logical form of 'Agnes is running or Agnes is not running' (or a variant using 'it is not true that...') is $A \vee \neg A$. Now, the answer, as throughout this chapter, depends on how you define *possible circumstance* and the truth conditions for negation and disjunction. On certain assumptions about the truth (and falsity) conditions for negations, the sentence (and, more broadly, sentence form) is logically true provided that disjunctions are true just if at least one disjunct is true. We give an answer along these lines. [NB: later chapters go into more examples of how these assumptions get rejected or modified.] *Proof.* Let c be any possible circumstance. We (here, though not in later chapters!) assume that the negation $\neg A$ is true-in-c iff A is false-in-c *if and only if* A is *not* true-in-c, and that $A \vee \neg A$ is true-in-c just if either A is true-in-c or $\neg A$ is true-in-c. Now, either A is true-in-c or it isn't. We show that either way $A \vee \neg A$ is true-in-c. Suppose, first, that A is true-in-c. Then, by said conditions on disjunction, $A \vee \neg A$ is true-in-c. Suppose, in turn, that A is not true-in-c. Then, by said conditions on negation (and falsity), A is false-in-c and, hence, $\neg A$ is true-in-c. Since $\neg A$ is true-in-c, conditions on disjunction give us (again) that $A \vee \neg A$ is true-in-c.

7. Consider the following argument.
   (a) Max is a bachelor.
   (b) Therefore, Max is unmarried.

   Neither sentence has any of our given connectives, and so both sentences are atomic, at least according to our definitions above. As such, atomics have no significant logical form. Instead, following the policy according to which distinct sentences are represented by distinct letters,[2] we would represent the argument form thus: $A \therefore B$. Is this argument form *valid*? If so, why? If not, why not? If there's not enough information to tell, what is the missing information? What premise might be added to make the argument valid?

   Answer *[GB]*. $A \therefore B$ is not a valid argument form. For a counterexample, it suffices to find two sentences and a possible circumstance c in which one of them is true and the other one is not true. For example, 2+2=4 and 2-2=4. Assuming that 'x is unmarried' expresses that x is *not* married, an additional premise that turns the example into a valid argument is: *either Max is a not bachelor or Max is unmarried.* The resulting argument form

[2] This is the policy that we will generally follow.

was proved valid in exercise 2.5. Another equivalent option (cf. p. 55), we may add: *if Max is a bachelor, then Max is unmarried.*

8. Consider, again, the argument above from 'Max is a bachelor' to 'Max is unmarried'. Is the conclusion a *necessary consequence* of the premise? If so, what, if anything, does this suggest about the role of 'logical form' in the 'necessary consequence' account of validity given in Chapter 1?

   Answer. Assuming that being a bachelor requires, inter alia, being unmarried, this is a valid argument on the necessary consequence account of validity. There is no possibility that Max is a bachelor yet not unmarried. Consequently, logical validity cannot exclusively be a matter of logical form in the necessary consequence account of validity—or, at least, so one logical theory might go (though we do not discuss this issue much in later chapters).

9. In your own words, say what it is to give *truth-in-a-case conditions* (or, for our purposes, truth conditions) for sentences. Why, if at all, is this activity—that is, giving so-called truth conditions—essential to an account of *logical consequence* as we've defined it (in Chapter 1)?

Sample answers

Answer 2. On the current 'possible circumstance' approach to cases, the argument form $A \wedge B \therefore B$ is valid iff there's no possible circumstance in which $A \wedge B$ is true but B not true (for any sentences A and B). On this account of validity, $A \wedge B \therefore B$ is valid. Proof: suppose that c is a possible circumstance in which $A \wedge B$ is true. By the (given) truth conditions for conjunction (see §**??**), if $A \wedge B$ is true-in-c then both A and B are true-in-c, and so B is true-in-c. But, by supposition, $A \wedge B$ *is* true-in-c, and so we conclude that B is true-in-c too. Hence, since what we've said about c applies to *any* possible circumstance, we conclude that there can't be any possible circumstance in which $A \wedge B$ is true but B not true; and, so, there can't be a counterexample to the given argument form.

Answer 3. Here are natural truth conditions (i.e., more accurately, truth-in-a-case conditions) for disjunction and negation.

- A disjunction $A \vee B$ is true-in-a-possible-circumstance-c if and only if A is true-in-c or B is true-in-c (or both).
- A negation $\neg A$ is true-in-a-possible-circumstance-c if and only if A is false-in-c.

3

Set-theoretic Tools

We provide examples of some answers to some problems.

Exercises

1. Write out $\mathcal{Y} \times \mathcal{Z}$ and $\mathcal{Z} \times \mathcal{Y}$, where $\mathcal{Y} = \{1, 2\}$ and $\mathcal{Z} = \{a, b, c\}$. Are $\mathcal{Y} \times \mathcal{Z}$ and $\mathcal{Z} \times \mathcal{Y}$ the same set? Justify your answer.

   Answer *[GB]*. $\mathcal{Y} \times \mathcal{Z} = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 1, c\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 2, c\rangle\}$ and $\mathcal{Z} \times \mathcal{Y} = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle c, 1\rangle, \langle c, 2\rangle\}$. Whether $\mathcal{Y} \times \mathcal{Z} = \mathcal{Z} \times \mathcal{Y}$ depends on the identities of $a, b, c, 1$ and 2. Consider two examples.

   (a) If we assume that a, b and c are all different from 1 and 2, then the two sets are different because $\langle 1, a\rangle \in \mathcal{Y} \times \mathcal{Z}$ but $\langle 1, a\rangle \notin \mathcal{Z} \times \mathcal{Y}$.

   (b) If $a = 1$ and $b = 2$ and $c = 1$, then, since $\mathcal{Z} = \{1, 2\}$, both sets $\mathcal{Y}$ and $\mathcal{Z}$ are identical, and so $\mathcal{Y} \times \mathcal{Z} = \mathcal{Z} \times \mathcal{Y} = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$.

   [We note, though, that it is not at all an unreasonable assumption that the letters (e.g., '*a*', '*b*', etc.) name items distinct from the numbers named by the numerals. If students make such an assumption, it's worth flagging that they're making it, but our opinion is that no major blunder has been made. Students *should*, of course, assume that the standard numerals are used for their standard objects—the numbers, etc.]

2. Using definition by abstraction, give brace-notation names (i.e., names formed using '{' and '}' as per the chapter) for each of the following sets.

   (a) The set of all even numbers.
   Answer. $\{x : x \text{ is an even number}\}$

   (b) The set of all felines.
   Answer. $\{x : x \text{ is a feline}\}$

   (c) The set of all tulips.
   Answer. $\{x : x \text{ is a tulip}\}$

   (d) The set of all possible worlds.
   Answer. $\{x : x \text{ is a possible world}\}$

   (e) The set of all people who love cats.
   Answer. $\{x : x \text{ is a person and } x \text{ loves cats}\}$

3. Assume that a, b, c, and d are distinct (i.e., non-identical) things. Which of the following relations are functions? (Also, if you weren't given that the various things are distinct, could you tell whether any of the following are functions? If so, why? If not, why not?)

   (a) $\{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$
   Answer. This is a function, and we could tell as much even without knowing that the given entities are distinct. Even if they were distinct, there still wouldn't be an entity in the relation that's related to two distinct items.

   (b) $\{\langle a, d\rangle, \langle b, d\rangle, \langle c, d\rangle, \langle d, d\rangle\}$
   Answer. This is a function, and for the same reason as in the first one (above), we don't need to know that the various entities are distinct.

   (c) $\{\langle a, b\rangle, \langle a, c\rangle, \langle b, d\rangle, \langle d, d\rangle\}$
   Answer. Given that $b \neq c$, this is not a function since a is related to both b and c. (If, contrary to the information in the problem, we had that $b = c$, then this would be a function.)

   (d) $\{\langle b, a\rangle, \langle c, d\rangle, \langle a, a\rangle, \langle b, d\rangle\}$
   Answer. This is not a function given that $a \neq d$ but b is related to both a and d. (Again, if, contrary to given information, we had that $a = d$, then this would be a function—indeed, the same function mentioned exercise 3b above.)

   (e) $\{\langle d, d\rangle, \langle d, b\rangle, \langle b, d\rangle, \langle a, d\rangle\}$
   Answer. This is not a function, since d is related to both d and b but $d \neq b$. (If, contrary to given information, we had that $d = b$, then this would be a function—a constant function mapping everything in $\{a, b\}$ to b. And we know that this is a function regardless of a's identity.)

4. Consider the relation of *biological motherhood*, which holds between objects x and y if and only if y is the biological mother of x. Is this relation a function? Justify your answer.

   Answer. Yes, this is a function because nothing has more one *biological* mother. (Of course, many people—and non-human animals, generally—have more than one mother in various familiar senses of 'mother'. But the question concerns the *biological* sense.)

5. Consider the relation of *loves*, which holds between objects x and y if and only if x loves y. Is this relation a function? Justify your answer.

<u>*Answer*</u>. The relation of *loves* is not a function. That it isn't a function is established by the fact that many people (and animals more generally) love more than one thing. (E.g., we—authors of this supplement—love the outdoors, philosophy, logic, and quite a few people.)

6. Since functions are relations, and all relations have a domain and range, it follows that functions have a domain and range. We say that the *domain* of a function f is the set of f's arguments (or 'inputs'), and the *range* of f is the set of f's values (or 'outputs'). Let the domain of g be $\{1, 2, 3\}$, where g is defined as follows.

$$g(x) = x + 22$$

   What is the range of g?
   <u>*Answer*</u>. The range of g is $\{g(1), g(2), g(3)\}$, which is the same as $\{23, 24, 25\}$.

7. Let $\mathcal{X} = \{1, 2\}$ and $\mathcal{Y} = \{\text{Max, Agnes}\}$. Specify *all* (non-empty) functions whose domain is $\mathcal{X}$ and range is $\mathcal{Y}$.
   <u>*Answer*</u> *[TR]*. There are exactly two functions with domain $\{1, 2\}$ and range $\{Max, Agnes\}$, namely, $\{\langle 1, Max\rangle, \langle 2, Agnes\rangle\}$ and $\{\langle 1, Agnes\rangle, \langle 2, Max\rangle\}$. [NB: we should note that $\{\langle 1, Max\rangle, \langle 2, Max\rangle\}$ is a function, but its range is $\{Max\}$, not $\{Max, Agnes\}$. Similarly $\{\langle 1, Agnes\rangle, \langle 2, Agnes\rangle\}$ is a function with an incorrect range. (If we let the *codomain* of a function simply be any set that includes, as a subset, the range of the function, then these other two functions would be functions from the given domain into the given *codomain* $\{Max, Agnes\}$. For purposes of this book, we rely just on the idea of a *range* for functions—viz., the set of values or 'outputs'.]

8. Specify all (non-empty) subsets of $\{1, 2, 3\}$.
   <u>*Answer*</u>. The non-empty subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$.

9. Show why each of the following are true for any sets $\mathcal{X}$ and $\mathcal{Y}$.
   (a) If $\mathcal{X} \neq \mathcal{Y}$, then $\mathcal{X} \cap \mathcal{Y} \subset \mathcal{X} \cup \mathcal{Y}$.
   <u>*Answer*</u>. We have to show that if $\mathcal{X} \neq \mathcal{Y}$, then the intersection of $\mathcal{X}$ and $\mathcal{Y}$ is a proper subset of the union of $\mathcal{X}$ and $\mathcal{Y}$. So, towards conditional proof, we assume that $\mathcal{X} \neq \mathcal{Y}$ (which tells us that there's some difference or other in membership). Now, to show the proper-subset claim, we first need to show that $\mathcal{X} \cap \mathcal{Y}$ is a subset of $\mathcal{X} \cup \mathcal{Y}$. If the former is empty, then the claim is easy (because vacuously true). So, let z be any element in $\mathcal{X} \cap \mathcal{Y}$. We need to show that $z \in \mathcal{X} \cup \mathcal{Y}$. But this follows immediately from the fact that $z \in \mathcal{X} \cap \mathcal{Y}$, and hence $z \in \mathcal{X}$ and $z \in \mathcal{Y}$, and hence z is in at least one of $\mathcal{X}$ and $\mathcal{Y}$, and hence $z \in \mathcal{X} \cup \mathcal{Y}$. So, $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{X} \cup \mathcal{Y}$. Towards *proper* subsethood,

we need to show is that there's something $u \in \mathcal{X} \cup \mathcal{Y}$ that's not in $\mathcal{X} \cap \mathcal{Y}$. This, finally, is where our assumption about $\mathcal{X} \neq \mathcal{Y}$ comes into play. In particular, we have from our initial (antecedent) assumption that $\mathcal{X} \neq \mathcal{Y}$. Suppose, for reductio, that there's nothing in $\mathcal{X} \cup \mathcal{Y}$ that isn't in $\mathcal{X} \cap \mathcal{Y}$, that is, that $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{Y} \cup \mathcal{Y}$. Well, then, since we just proved above that $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{X} \cup \mathcal{Y}$, we conclude that $\mathcal{X} \cup \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}$. But this can only happy if $\mathcal{X} = \mathcal{Y}$,[1] which contradicts our assumption that $\mathcal{X} = \mathcal{Y}$.

(b) If $\mathcal{X} \subset \mathcal{Y}$, then $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$.
Answer. This is sufficiently answered in Sample Answers.

(c) If $\mathcal{X} \subset \mathcal{Y}$, then $\mathcal{X} \cap \mathcal{Y} \subset \mathcal{Y}$.
Answer. Suppose, for conditional proof, tat $\mathcal{X} \subset \mathcal{Y}$. We need to show that $\mathcal{X} \cap \mathcal{Y}$ is a proper subset of $\mathcal{Y}$. We first show that it's a subset. Towards that end, let $z \in \mathcal{X} \cap \mathcal{Y}$, in which case, by definition of intersection, we have that $z \in \mathcal{X}$ and $z \in \mathcal{Y}$. Hence, $z \in \mathcal{Y}$. So, $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{Y}$, by definition of subsethood. Towards showing *properness*, we suppose, for reductio, that there is nothing in $\mathcal{Y}$ that isn't also in $\mathcal{X} \cap \mathcal{Y}$, that is, we suppose that $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{Y}$. Since we already have that $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{Y}$, we have that $\mathcal{X} \cap \mathcal{Y} = \mathcal{Y}$. But here is where our initial (antecedent) assumption comes into play. We already have that $\mathcal{X} \subset \mathcal{Y}$, and so there's something (say, z) that's in $\mathcal{Y}$ but not in $\mathcal{X}$. Hence, $z \in \mathcal{Y}$, but $z \notin \mathcal{X}$, and since not in $\mathcal{X}$, we have (by definition of intersection) that $z \notin \mathcal{X} \cap \mathcal{Y}$. But this contradicts that $\mathcal{X} \cap \mathcal{Y} = \mathcal{Y}$. Hence, we conclude that our assumption for reductio is incorrect, and so conclude that $\mathcal{X} \cap \mathcal{Y}$ is not only a subset of $\mathcal{Y}$, but a proper one (given the antecedent assumption that $\mathcal{X}$ is a proper subset of $\mathcal{Y}$).

10. Let f be some function with $dom(f) = \mathcal{X}$ (i.e., the domain of f is $\mathcal{X}$), for some arbitrary (non-empty) set $\mathcal{X}$. We say that our function f is a function *from* $\mathcal{X}$ *into* $\mathcal{Y}$ if $ran(f) \subseteq \mathcal{Y}$. Given this terminology, specify *all* (non-empty) functions from $\{A, B\}$ *into* $\{1, 2, 3\}$, where A and B are distinct sentences. (Note that any such function must map *every* element of the domain to something in $\{1, 2, 3\}$.)

    Answer [TR]. Let the notation $f : \mathcal{X} \longrightarrow \mathcal{Y}$ represent a relation f with $dom(f) = \mathcal{X}$ and $ran(f) = \mathcal{Y}$. Then the non-empty functions from $\{A, B\}$ into $\{1, 2, 3\}$ can be categorized by the non-empty subsets of $\{1, 2, 3\}$ (see exercise 8) as follows.

[1] Proof: suppose that $\mathcal{X} \cup \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}$ but, for reductio, also $\mathcal{X} \neq \mathcal{Y}$. Then there's a difference in membership, and so, without loss of generality, let's say that there's something in $\mathcal{Y}$ that isn't in $\mathcal{X}$, and let's let the item be z. Since $z \in \mathcal{Y}$, then $z \in \mathcal{X} \cup \mathcal{Y}$. But $z \notin \mathcal{X}$, and hence $z \notin \mathcal{X} \cap \mathcal{Y}$. Contradiction.

(a) All functions $f : \{A, B\} \longrightarrow \{1\}$, namely, $\{\langle A, 1\rangle, \langle B, 1\rangle\}$.
(b) All functions $f : \{A, B\} \longrightarrow \{2\}$, namely, $\{\langle A, 2\rangle, \langle B, 2\rangle\}$.
(c) All functions $f : \{A, B\} \longrightarrow \{3\}$, namely, $\{\langle A, 3\rangle, \langle B, 3\rangle\}$.
(d) All functions $f : \{A, B\} \longrightarrow \{1, 2\}$ not given above: $\{\langle A, 1\rangle, \langle B, 2\rangle\}$ and $\{\langle A, 2\rangle, \langle B, 1\rangle\}$. [NB: functions in 10a and 10b above count here too.]
(e) All functions $f : \{A, B\} \longrightarrow \{1, 3\}$ not given above: $\{\langle A, 1\rangle, \langle B, 3\rangle\}$ and $\{\langle A, 3\rangle, \langle B, 1\rangle\}$. [NB: functions in 10a and 10c above count here too.]
(f) All functions $f : \{A, B\} \longrightarrow \{2, 3\}$ not given above: $\{\langle A, 2\rangle, \langle B, 3\rangle\}$ and $\{\langle A, 3\rangle, \langle B, 2\rangle\}$. [NB: functions in 10b and 10c above count here too.]
(g) There are no functions $f : \{A, B\} \longrightarrow \{1, 2, 3\}$, only relations. (Recall that, on this notation, $ran(f) = \{1, 2, 3\}$.)

11. Let $\mathcal{X}$ be an arbitrary set and f an arbitrary function. We say that f is an *operator on* $\mathcal{X}$ if and only if the $dom(f) = \mathcal{X}$ and $ran(f) \subseteq \mathcal{X}$. Consider the following operator on $\{1, 0\}$.

$$g(x) = 1 - x$$

Now, imagine a function v that assigns either 1 or 0 to each atomic sentence of our language, so that, for any atomic sentence A of our language, we have it that $v(A) = 1$ or $v(A) = 0$. Answer the following questions.

(a) Suppose that $v(A) = 1$. What is $g(v(A))$?
Answer. This is sufficiently answered in Sample Answers.

(b) Suppose that $v(A) = 0$. What is $g(v(A))$?
Answer. $g(v(A)) = 1$ if $v(A) = 0$.

(c) If $v(A) = 1$, what is $g(g(v(A)))$?
Answer. $g(g(v(A))) = 1$ if $v(A) = 1$.

(d) Is it true that $g(g(x)) = 1$ just when $x = 1$?
Answer [GB]. Yes, $g(g(x)) = 1$ just if $x = 1$. *Proof.* $dom(g) = \{1, 0\}$, and so there are exactly two cases:
  i. if $x = 1$, then $g(g(x)) = g(g(1)) = g(1 - 1) = g(0) = 1 - 0 = 1$;
  ii. if $x = 0$, then $g(g(x)) = g(g(0)) = g(1 - 0) = g(1) = 1 - 1 = 0$.

(e) How, *if at all*, is the given function g similar to negation (as you thought about it in Chapter 2)?
Answer [GB]. The operator g turns 0 into 1 and 1 into 0. If we correlate 0 with falsity and 1 with truth, then negation behaves similar to

g if negation is a function such that the negation of a true sentence is false and the negation of a false sentence true. However, in places in Chapter 2 (e.g., solutions to exercise 3), and certainly in subsequent chapters, we assume only the latter, but not the former. (Exactly how negation behaves is a rich part of logical theorizing!)

Sample answers

Here are some sample answers. (In the first one, the answer is somewhat involved for purposes of illustrating, in a fairly step-by-step fashion, how one might go about proving the given claims.)

Answer 9b. We have to show that if $\mathcal{X} \subset \mathcal{Y}$, then $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$. We show this (viz., the given conditional) by so-called conditional proof: we assume that the antecedent is true (viz., that $\mathcal{X} \subset \mathcal{Y}$), and then show—via valid steps (!)—that the consequent is true. (Usually, we do this simply by invoking definitions involved.) So, suppose that $\mathcal{X} \subset \mathcal{Y}$, in which case, by definition of *proper subset* (see Def. 11), it follows that anything in $\mathcal{X}$ is in $\mathcal{Y}$, and that $\mathcal{Y}$ contains something that $\mathcal{X}$ doesn't contain. Now, we need to show the consequent of our target conditional: viz., that $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$. This is an identity claim: it claims that the two given sets are identical. How do we show that they're identical? Well, we have to invoke the definition of identity for sets, which tells us that, in this case, $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$ iff both $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{Y}$ contain exactly the same things. In other words, we show that $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$ by showing that something (no matter what it is) is in $\mathcal{X} \cup \mathcal{Y}$ if and only if it's in $\mathcal{Y}$. So, in effect, we have to show that two different conditionals are true to show that the two sets are identical:

9b.1 If something (no matter what it is) is in $\mathcal{X} \cup \mathcal{Y}$, it is in $\mathcal{Y}$.

9b.2 If something (no matter what it is) is in $\mathcal{Y}$, it is in $\mathcal{X} \cup \mathcal{Y}$.

And here, we can just do so-called conditional proofs again for each of (9b.1) and (9b.2): we assume the given antecedents and show, via valid steps (usually just appealing to the definitions), that the given consequents follow. So, for (9b.1), we assume that something—call it (no matter what it is) 'z'—is in $\mathcal{X} \cup \mathcal{Y}$. What we have to show is that z is in $\mathcal{Y}$. Well, by assumption, we have that $z \in \mathcal{X} \cup \mathcal{Y}$, in which case, *by definition of union* (see Def. **??**), if $z \in \mathcal{X} \cup \mathcal{Y}$ then $z \in \mathcal{X}$ *or* $z \in \mathcal{Y}$. In the latter case, we have what we want (viz., that $z \in \mathcal{Y}$). What about the former case in which $z \in \mathcal{X}$? Do we also get that $z \in \mathcal{Y}$? Yes: we get this from our initial supposition that $\mathcal{X} \subset \mathcal{Y}$, which assures that anything in $\mathcal{X}$ is in $\mathcal{Y}$. What this tells us is that, either way, if something z (no matter what z may be) is in $\mathcal{X} \cup \mathcal{Y}$, then it's also in $\mathcal{Y}$ (provided that, as we've assumed from the start, $\mathcal{X} \subset \mathcal{Y}$). And this is what we wanted to show for (9b.1).

With respect to (9b.2), we assume that something z (no matter what z is) is in $\mathcal{Y}$. We need to show that $z \in \mathcal{X} \cup \mathcal{Y}$. But this follows immediately from the definition of union (see Def. **??**).[2] According to the definition, something is in $\mathcal{X} \cup \mathcal{Y}$ if and only if it's either in $\mathcal{X}$ or in $\mathcal{Y}$. Hence, given that (by supposition) $z \in \mathcal{Y}$, we have it that $z \in \mathcal{X} \cup \mathcal{Y}$.

Taking stock of Answer 9.b. What we've proved, in showing (9b.1) and (9b.2), is that, under our assumption that $\mathcal{X} \subset \mathcal{Y}$, something (no matter what it is) is in $\mathcal{X} \cup \mathcal{Y}$ iff it's in $\mathcal{Y}$. By definition of identity for sets (see Def. **??**), this tells us that, under our assumption that $\mathcal{X} \subset \mathcal{Y}$, the sets $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{Y}$ are identical. And this is what (9b) required us to show.

Answer 11a. If $v(A)$ is 1, then, plugging 1 in for x in the definition of function g, we have that $g(1) = 1 - 1$, and so $g(v(A))$ is 0.

[2]Well, we're assuming that so-called Addition is valid, that is, that a disjunction is implied by either of its disjuncts. Some logical theories question this (see, e.g., Chapter 12 in which one such theory is briefly waved at); however, we'll assume it in our reasoning throughout the book.

PART II

BASIC CONNECTIVES

4

Classical Theory

Exercises

1. Show that, on the classical theory, $A \therefore \neg\neg A$ is valid.

   <u>*Answer*</u>. One way of answering this uses reasoning as in §4.3 (see the proof that $\neg\neg A \therefore A$ is classically valid). For purposes of illustration, we here use the 'formal picture', explicitly invoking our functions (or 'valuations') $v : \mathcal{S} \longrightarrow \mathcal{V}$. In particular, let ν be (or represent, if you prefer) any classical case in which A is true, that is, $\nu(A) = 1$. Now, since $\nu(A) = 1$, we have that, since ν is a function, $\nu(A) \neq 0$. In turn, the classical 'truth conditions' for negation tell us that $\nu(A) \neq 0$ iff $\nu(\neg A) \neq 1$. So, we have that $\nu(\neg A) \neq 1$. But since ν assigns *every* sentence either 1 or 0, we conclude that $\nu(\neg A) = 0$ since $\nu(\neg A) \neq 1$. But, then, by the clause for negation, $\nu(\neg\neg A) = 1$ since $\nu(\neg A) = 0$. Hence, since this reasoning applies to any classical case (or representation of a classical case) ν, we conclude that there's no classical case in which $\nu(A) = 1$ and $\neg\neg A) \neq 1$, that is, no classical case in which A is true but $\neg\neg A$ untrue. So, $A \therefore \neg\neg A$ is valid according to the classical theory.

2. Show that, according to the classical theory, $A, B \therefore A \wedge B$ is valid.

   <u>*Answer*</u>. Again, we can show this in various ways. Invoking the 'formal story' as in the answer above, we can reason as follows. Suppose that each of A and B are true according to some classical case ν, that is, that $\nu(A) = 1 = \nu(B)$. By the truth conditions (i.e., truth-in-a-case conditions) for conjunction, it follows immediately that $\nu(A \wedge B) = 1$, that is, that $A \wedge B$ is true in the given case. Hence, since this reasoning applies to any classical case, we conclude that there is no classical case in which both A and B are true but $A \wedge B$ untrue.

3. In addition to our definition of *logical truth* (true-in-*every case*), let us define *contingent* and *logically false* as follows.

   - Sentence A is *logically false* iff it is false-in-*every case*.
   - Sentence A is *contingent* iff it is true-in-*some case*, and false-in-*some case*.

For each of the following sentences of $\mathcal{L}$, say whether, according to the classical theory, it is logically true, logically false, or contingent.[1]

(a) $p \rightarrow p$
<u>*Answer*</u>. Status: logically true.

(b) $p \rightarrow \neg p$
<u>*Answer*</u>. Status: contingent. Any case in which p is true is a case in which $p \rightarrow \neg p$ is false, and any case in which p is false is one in which $p \rightarrow \neg p$ is true. There are, of course, classical cases of each sort. [NB: it may be useful for your students to recall the primitive notation in terms of negation and disjunction: $A \rightarrow B$ abbreviates $\neg A \vee B$, and so $p \rightarrow \neg p$ abbreviates $\neg p \vee \neg p$.]

(c) $p \wedge \neg p$
<u>*Answer*</u>. Status: logically false.

(d) $q \vee p$
<u>*Answer*</u>. Status: contingent.

(e) $q \wedge (p \vee q)$
<u>*Answer*</u>. Status: contingent.

(f) $q \vee (p \wedge q)$
<u>*Answer*</u>. Status: contingent.

(g) $q \leftrightarrow \neg p$
<u>*Answer*</u>. Status: contingent.

(h) $(p \wedge (p \rightarrow q)) \rightarrow q$
<u>*Answer*</u>. Status: logically true.

4. For each of the valid forms in §4.7, give a proof that they're valid. (Carefully consider whether there can be a classical case in which the premises are true and the conclusion false. To do this, you'll need to keep going back to the truth conditions for the various connectives. One useful method for doing this is called *Reductio*. The idea, in this context, is to assume that there *is* a counterexample to the given argument, that is, that there *is* a classical case v that satisfies the premises but assigns 0 to the conclusion. If this assumption leads to a contradiction—in particular, that some sentence gets assigned both 1 and 0, which is impossible—you conclude, via Reductio,

[1] Again, for convenience, otherwise requisite parentheses are dropped when confusion won't arise.

that the initial assumption was wrong, that is, that there can't, contrary to your initial assumption, be a classical counterexample.)

Answer. In our answers here (in the supplement), we use the 'formal picture', but one could also do this using the slightly less formal notation (involving $\models_1$ and the like) to which we return fully in Chapters 8–12. We often use 'reductio reasoning' here (in this supplement), and it may be useful—and, in these examples, is often just as useful (and more illuminating)—to do some direct proofs with your students. Also, importantly, here and in subsequent chapters we rely on the fact that, in our target logic, we have that $A \rightarrow B$ is true just if A is false or B is true, and false just if A is true and B false.

- Modus Ponens: $A \rightarrow B, A \vdash B$
  Answer. Let v be any (formal representation of a) classical case, and let $v(A) = v(A \rightarrow B) = 1$. By truth conditions for the arrow, we have that $v(A) = 0$ or $v(B) = 1$. Since, by supposition, $v(A) = 1$, we conclude that $v(A) \neq 0$ and, hence, that $v(B) = 1$. Hence, since this reasoning applies to any classical case, we conclude that there's no such case in which both A and $A \rightarrow B$ are true but B untrue.

- Modus Tollens: $A \rightarrow B, \neg B \vdash \neg A$
  Answer. Suppose, for reductio, that there's a classical case v such that $v(A \rightarrow B) = 1 = v(\neg B)$ but $v(\neg A) = 0$. By negation conditions, $v(A) = 1$ and $v(B) = 0$. But, then, $v(A \rightarrow B) = 0$, which contradicts our initial supposition. Hence, there can't be such a case.

- Disjunctive Syllogism: $A \vee B, \neg A \vdash B$
  Answer. Assume that $v(A \vee B) = 1 = v(\neg A)$. By truth conditions for negation, we have that $v(A) = 0$. But, then, since $v(A \vee B) = 1$, the truth conditions for disjunction imply that $v(B) = 1$.

- Contraposition: $A \rightarrow B \dashv\vdash \neg B \rightarrow \neg A$
  Answer [LRD]. Suppose that $v(A \rightarrow B) = 1$. For reductio, suppose that $v(\neg B \rightarrow \neg A) = 0$, in which case, by conditions on the arrow, $v(\neg B) = 1$ and $v(\neg A) = 0$, and so, by negation conditions, $v(A) = 1$ and $v(B) = 0$, in which case, by the arrow's conditions, $v(A \rightarrow B) = 0$. This contradicts our initial supposition.
  Answer [RLD]. Conversely, suppose that $v(\neg B \rightarrow \neg A) = 1$, but, for reductio, $v(A \rightarrow B) \neq 1$, in which case $v(A) = 1$ and $v(B) = 0$, in which case $v(\neg A) = 0$ and $v(\neg B) = 1$, in which case $v(\neg B \rightarrow \neg A) = 0$, which contradicts the initial supposition.

- Explosion: $A, \neg A \vdash B$

Answer. A *counterexample* is a case in which *all* premises (if any) are true and the conclusion untrue. Since, according to the classical theory, there's no case in which both A and $\neg A$ are true, we cannot have a counterexample to explosion. Hence, the argument form (viz., explosion) winds up being valid according to the classical theory.

- Addition: $A \vdash A \vee B$
  Answer. The truth conditions for disjunction tell us that $A \vee B$ is true-in-a-case just if at least one disjunct is true-in-that-case. Hence, there can't be a case c such that $c \models_1 A$ but $c \not\models_1 A \vee B$. So, the argument form is valid, according to the classical theory.

- Adjunction: $A, B \vdash A \wedge B$
  Answer. The truth conditions for conjunction tell us that $A \wedge B$ is true-in-a-case just if both conjuncts are true-in-that-case. Hence, there can't be a case c such that each of A and B is true-in-c but $A \wedge B$ is not true-in-c. So, the argument form is valid, according to the paracomplete theory.

- Simplification: $A \wedge B \vdash A$
  Answer. The truth conditions for conjunction tell us that $A \wedge B$ is true-in-a-case just if both conjuncts are true-in-that-case. So, any case in which $A \wedge B$ is true is one in which A is true, given the running truth conditions.

- De Morgan: $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$
  Answer *[LRD]*. Let $v(\neg(A \vee B)) = 1$, in which case $v(A \vee B) = 0$, in which case $v(A) = 0 = v(B)$, and so $v(\neg A) = 1 = v(\neg B)$, and so $v(\neg A \wedge \neg B) = 1$. (The truth/falsity conditions invoked are negation, disjunction, negation, and then conjunction, in that order.)
  Answer *[RLD]*. Conversely, suppose that $v(\neg A \wedge \neg B) = 1$, in which case $v(\neg A) = 1 = v(\neg B)$, in which case $v(A) = 0 = v(B)$, and so $v(A \vee B) = 0$ by the truth (falsity) conditions for disjunction. But, now, by the truth conditions for negation, $v(\neg(A \vee B)) = 1$.

- De Morgan: $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$
  Answer *[LRD]*. Suppose that $v(\neg(A \wedge B) = 1$, in which case $v(A \wedge B) = 0$. So, by falsity conditions for conjunctions, one of A and B is false-in-v, in which case one of $\neg A$ and $\neg B$ is true-in-v, and so, by disjunction conditions, we get that $v(\neg A \vee \neg B) = 1$.
  Answer *[RLD]*. Suppose that $v(\neg A \vee \neg B) = 1$, in which case either $v(\neg A) = 1$ or $v(\neg B) = 1$, and so either $v(A) = 0$ or $v(B) = 0$. Either way, at least one conjunct in $A \wedge B$ is false-in-v, and so $v(A \wedge B) = 0$,

and hence $v(\neg(A \wedge B)) = 1$.

- Double Negation: $\neg\neg A \dashv\vdash_{K3} A$
  Answer. We can do both directions in one go by relying on the truth (falsity) conditions for negation. In particular, $v(\neg\neg A) = 1$ iff $v(\neg A) = 0$ iff $v(A) = 1$. Hence, if you've got a case v in which $\neg\neg A$ is 1, then that's a case in which A is 1 too, and vice versa.

5. Prove that, where $\vdash$ is our basic classical consequence relation, each of the following are true (i.e., that the given argument forms are valid in the basic classical theory).

   (a) $A \to B, B \to C \vdash A \to C$.
   Answer. Suppose that $v(A \to C) = 0$, in which case $v(A) = 1$ and $v(C) = 0$. But, then, in order for $A \to B$ to be true-in-v, B must be true-in-v, in which case $B \to C$ isn't true-in-v. So, there can't be a (classical) case in which $A \to C$ is untrue without one of the premises being untrue. Hence, there can't be a (classical) counterexample to the argument form.

   (b) $(A \vee B) \wedge C, A \to \neg C \vdash B$.
   Answer. Suppose, for reductio, $v(B) = 0$ but both premises true, in which case $v(A \to \neg C) = 1$, and so $v(\neg A \vee \neg C) = 1$, and so either $v(A) = 0$ or $v(C) = 0$. Either way, $(A \vee B) \wedge C$ can't be true-in-v.

   (c) $(A \vee B) \wedge C \dashv\vdash (A \wedge C) \vee (B \wedge C)$.
   Answer *[LRD]*. Let $v((A \vee B) \wedge C) = 1$ but, for reductio, let the conclusion be false, in which case $v(A \wedge C) = 0$ and $v(B \wedge C) = 0$. Since, by initial supposition, $v(C) = 1$, it must be that $v(A) = 0 = v(B)$. But, then, $v(A \vee B) = 0$, contradicting our initial supposition (which requires that it be true).
   Answer *[RLD]*. Conversely, let $v((A \wedge C) \vee (B \wedge C)) = 1$, in which case either C and B are true-in-v or C and A are true-in-v. Either way, we have, via disjunction conditions, that $A \vee B$ is true-in-v in addition to C's being true-in-v.

   (d) $(A \wedge B) \vee C \dashv\vdash (A \vee C) \wedge (B \vee C)$.
   Answer. The answer here is similar to that above.

   (e) $A \to B, \neg A \to B \vdash B$.
   Answer. Suppose that $v(A \to B) = v(\neg A \to B)$, in which case, $v(\neg A \vee B) = v(\neg\neg A \vee B) = 1$. Since we cannot have $v(\neg A) = v(\neg\neg A) = 1$, it must be that $v(B) = 1$.

6. Suppose that, instead of functions, we model our classical cases as *sets of sentences*. A *case*, on this approach, is a set $\mathcal{X}$ of $\mathcal{L}$ sentences. In turn, we say that *truth in a case* is just membership—i.e., being an element—in such a set. Your task is two-fold:
   (a) What constraints do we impose on the given cases for them to be *classical*—i.e., 'complete' and 'consistent'?
   Answer. We simply say that for any such (new sort of) case $\mathcal{X}$, we have that exactly one of A and $\neg A$ is in $\mathcal{X}$, for all sentences A of $\mathcal{L}$.

   (b) What are the truth conditions for conjunctions, disjunctions, and negations on this approach?
   Answer.
      * $\neg A$ is in $\mathcal{X}$ iff A is *not* in $\mathcal{X}$.
      * $A \wedge B$ is in $\mathcal{X}$ iff both A and B are in $\mathcal{X}$.
      * $A \vee B$ is in $\mathcal{X}$ iff at least one of A and B is in $\mathcal{X}$.

7. Can you think of a way of defining $\vee$ in terms of $\neg$ and $\wedge$? (Hint: see whether you can come up with a sentence that uses only $\neg$ and $\wedge$ but has exactly the same 'truth table' as $\vee$.) If so, you've shown that, strictly speaking, we can reduce our number of basic connectives to just $\neg$ and $\wedge$ (and treat $\vee$, like the others, as defined).
   Answer. Define $A \vee B$ as $\neg(\neg A \wedge \neg B)$.

8. Related to the previous question, can you think of a way of defining $\wedge$ in terms of $\vee$ and $\neg$?
   Answer. Define $A \wedge B$ as $\neg(\neg A \vee \neg B)$.

Sample answers

Answer 3b. The sentence $p \to \neg p$ is contingent: it is true-in-*some case* and false-in-*some case*. Proof: p is atomic, and so there are cases in which p is true, and also cases in which p is false. Let v be any case in which p is true, that is, $v(p) = 1$. By the classical treatment of negation, $v(\neg p) = 0$. By definition, $p \to \neg p$ is equivalent to $\neg p \vee \neg p$.[2] By the truth conditions for disjunction, $\neg p \vee \neg p$ is true iff one of its disjuncts is true; but $\neg p$ is the only disjunct, and it is not true-in-the-given-case, since $v(\neg p) = 0$. So, v is a case in which $p \to \neg p$ is not true. On the other hand, consider any case v' in which p is false, that is, $v'(p) = 0$. By the truth conditions for negation, $v'(\neg p) = 1$, in which case, by the truth conditions for disjunction, $v'(\neg p \vee \neg p) = 1$, and hence $v'(p \to \neg p) = 1$. So, v' is a case in which $p \to \neg p$ is true. Hence, there are cases in which $p \to \neg p$ is true and cases in which it is false.

Answer 4-LEM. To see that LEM is a valid form (i.e., that all of its instances are logically true sentences), we need to show that there's no case in which $A \vee \neg A$

[2]Recall from §4.6 that $A \to B$ is defined to be $\neg A \vee B$.

is false (for any sentence A). We do this by Reductio. Suppose, for reductio, that there's some case v such that $v(A \vee \neg A) = 0$ (for some sentence A). The truth conditions for disjunction tell us that $v(A \vee \neg A) = 1$ if and only if $v(A) = 1$ or $v(\neg A) = 1$. Since, by supposition, $v(A \vee \neg A) \neq 1$ (since v is a function which has assigned 0 to $A \vee \neg A$), we have it that $v(A) \neq 1$ and $v(\neg A) \neq 1$. But since v has to assign either 1 or 0 to every sentence, we conclude that $v(A) = 0$ and $v(\neg A) = 0$. But this is impossible, since, by truth conditions for negation, $v(\neg A) = 1$ iff $v(A) = 0$. So, we conclude that our initial supposition—namely, that there's some case v in which $A \vee \neg A$ (for some A) is false—is itself untrue. Hence, we conclude that there cannot be a (classical) case in which $A \vee \neg A$ (for some A) is false, which is to say that LEM is valid.

Answer 4-Simplification. To see that $A \wedge B$ implies A in the classical theory, we can use Reductio.[3] Suppose, for reductio, that there's a counterexample to $A \wedge B \therefore A$, that there's some (classical) case v such that $v(A \wedge B) = 1$ but $v(A) = 0$. The truth conditions for conjunction tell us that $v(A \wedge B) = 1$ iff $v(A) = 1$ and $v(B) = 1$. But, then, we have it that $v(A) = 1$, since (by supposition) we have it that $v(A \wedge B) = 1$. But, by supposition, we also have it that $v(A) = 0$. This is impossible, since v is a function and, so, cannot assign anything to *both* 1 and 0. (If you've forgotten the chief feature of functions, you should turn back to Chapter 3 for a quick review!) Hence, we reject our initial assumption that there's a counterexample to Simplification, and conclude that there's no counterexample—and, hence, that the given form is valid.

[3]NB: we certainly do not need to use Reductio, since the answer falls directly out of the truth conditions for conjunction; however, it may be useful to give a few examples of Reductio reasoning.

5

A Paracomplete Theory

Exercises

1. Answer any questions raised in the text.
2. We noted that Excluded Middle is not a logically true (sentence) form in the paracomplete theory. Question: is there any case in which $A \vee \neg A$ is *false*? If so, give such a case. If not, say why not.

   Answer. There is no case in which $A \vee \neg A$ is false, that is, no case c such that $c \models_0 A \vee \neg A$. There are various ways to see this, but the 'formal picture' makes it plain. Suppose, for reductio, that $v(A \vee \neg A) = 0$ for some paracomplete case (or, if you want, some formal representation of a paracomplete case) v. But truth (or, in this case, falsity) conditions for disjunction, we have it that $v(A) = 0 = v(\neg A)$. But by truth (falsity) conditions for negation, $v(\neg A) = 0$ iff $v(A) = 1$, and so $v(A) = 1$ and, as we supposed, $v(A) = 0$. This is impossible, given that v is a function (and, hence, cannot assign two different values to the same argument). Hence, we conclude that there's no case in which $A \vee \neg A$ is false on our given paracomplete theory.

3. Recall, from Chapter 4, our definitions of *logically true, logically false*, and *contingent*, where A is any sentence.
   - A is *logically true* iff it is true-in-*every case*.
   - A is *logically false* iff it is false-in-*every case*.
   - A is *contingent* iff it is true-in-*some case*, and false-in-*some case*.

   Our given paracomplete theory, as above, has no logical truths. Give a (paracomplete) counterexample to each of the following sentences (i.e., a paracomplete case in which the sentence is *not true*). In addition, specify which, if any, of the following sentences are logically false, and which are contingent.

   (a) $p \rightarrow p$

   Answer. Counterexample: let $v(p) = \mathsf{n}$. Status: neither logically false nor contingent. [*Note*. See exercise 4 below.]

   (b) $p \rightarrow \neg p$

   Answer. Counterexample: let $v(p) \in \{1, \mathsf{n}\}$. Status: contingent.

   (c) $p \wedge \neg p$

<u>*Answer*</u>. Counterexample: let $v(p) \in \{1, \mathsf{n}, 0\}$. Status: neither logically false nor contingent. [*Note*. If we defined the notion of *logically untrue* as untrue-in-all-cases, then this sentence would be logically untrue.]

(d) $q \vee p$
<u>*Answer*</u>. Counterexample: let $v(p) \in \{\mathsf{n}, 0\}$ and $v(q) \in \{\mathsf{n}, 0\}$. Status: contingent.

(e) $q \wedge (p \vee q)$
<u>*Answer*</u>. Counterexample: let $v(q) \in \{\mathsf{n}, 0\}$. Status: contingent.

(f) $q \vee (p \wedge q)$
<u>*Answer*</u>. Counterexample: let $v(q) \in \{\mathsf{n}, 0\}$. Status: contingent.

(g) $q \leftrightarrow \neg p$
<u>*Answer*</u>. Counterexample: let $v(q) = v(p)$ or let at least one of $v(q)$ and $v(p)$ be n. Status: contingent.

(h) $(p \wedge (p \rightarrow q)) \rightarrow q$
<u>*Answer*</u>. Counterexample: let $v(q) = \mathsf{n}$ and $v(p) \in \{1, \mathsf{n}\}$. Status: not contingent, since never false-in-a-case—and, so, not logically false.

(i) $p \vee \neg p$
<u>*Answer*</u>. Counterexample: let $v(p) = \mathsf{n}$. Status: not contingent, since never false-in-a-case—and, so, not logically false.

(j) $\neg(p \wedge \neg p)$
<u>*Answer*</u>. Counterexample: let $v(p) = \mathsf{n}$. Status: not contingent, since never false-in-a-case—and, so, not logically false.

4. Suppose that we define a different, broader sort of 'contingency' thus:

- A sentence A is *broadly contingent* iff it is true-in-*some case* and *not* true-in-*some case* (i.e., *untrue*-in-some case).

Which, if any, of the displayed sentences (3a)–(3j), from exercise 3, are broadly contingent? Also: in the given paracomplete theory, can a sentence be broadly contingent without being contingent?

<u>*Answer*</u>. Except for (3c), all of (3a)–(3j) are broadly contingent. That, on the given paracomplete theory, some sentences can be broadly contingent without being contingent is a consequence of exercises 3 and 4 together.

5. By way of contrast, redo exercises 3 and 4 above in terms of the *classical* theory.

Answer. We take each part of the exercise in reverse (treating exercise 4 first).

(a) We note that the 'distinction' between *contingent* and *broadly contingent* collapses in the classical theory, since, in that theory, there's no different between *falsity* and *untruth*.

(b) The classical theory agrees with the paracomplete one on all of the sentences classified as contingent in exercise 3. The difference comes with the non-contingent ones. In particular, the classical theory classifies (3a) and (3h)–(3j) as *logically true*, and classifies (3c) as *logically false*.

6. For each of the valid forms in §5.7, give a proof that they're valid.

Answer. In our answers here (in the supplement), we use the 'formal picture', but one could also do this using the slightly less formal notation (involving $\models_1$ and the like) to which we return fully in Chapters 8–12. We do not often use 'reductio reasoning' in these proofs, though it may be useful to give your students a sense of how reductio proofs might go. (In general, if you're trying to establish that X implies A in the paracomplete theory, then you can't just assume that, for some case v, X is satisfied but A is *false* in the case, that is, $v(A) = 0$. Instead, one needs to assume that X is satisfied but A *untrue*, which is a disjunction between being *false* or *gappy* (so to speak).) We also rely on the fact that if B is false-in-a-case (i.e., has value 0), then $A \rightarrow B$ is true in that case only if A is false in that case (i.e., has value 0). [It may worth having students see this point.]

- Modus Ponens: $A \rightarrow B, A \vdash_{K3} B$
  Answer. Let v be any (formal representation of a) paracomplete case, and let $v(A) = v(A \rightarrow B) = 1$, and so, by definition of the arrow, $v(\neg A \vee B) = 1$. By negation conditions, $v(\neg A) = 0$, and hence, by disjunction conditions, $v(\neg A \vee B) = 1$ only if $v(B) = 1$.

- Modus Tollens: $A \rightarrow B, \neg B \vdash_{K3} \neg A$
  Answer. Let $v(A \rightarrow B) = 1 = v(\neg B)$. In this case, $v(B) = 0$, and so $v(A \rightarrow B) = 1$ only if $v(A) = 0$, and so only if $v(\neg A) = 1$.

- Disjunctive Syllogism: $A \vee B, \neg A \vdash_{K3} B$
  Answer. Assume that $v(A \vee B) = 1 = v(\neg A)$, and so$v(A) = 0$. Since $v(A \vee B) = 1$, disjunction conditions imply that $v(B) = 1$.

- Contraposition: $A \rightarrow B \dashv\vdash_{K3} \neg B \rightarrow \neg A$
  Answer *[LRD]*. Let $v(A \rightarrow B) = 1$, and so $v(\neg A \vee B) = 1$, and so either $v(\neg A) = 1$ or $v(B) = 1$. Hence, either $v(\neg B) = 0$ or $v(\neg A) = 1$, and so either $v(\neg\neg B) = 1$ or $v(\neg A) = 1$, and so $v(\neg\neg B \vee \neg A) = 1$, and so, by definition of the arrow, $v(\neg B \rightarrow \neg A) = 1$.

<u>*Answer*</u> *[RLD]*. Let $v(\neg B \to \neg A) = 1$, and so $v(\neg\neg B \vee \neg A) = 1$, and so either $v(\neg A) = 1$ or $v(B) = 1$, and so $v(\neg A \vee B) = 1$, and so, by definition of the arrow, $v(A \to B) = 1$.

- Explosion: $A, \neg A \vdash_{K3} B$
  <u>*Answer*</u>. A *counterexample* is a case in which *all* premises (if any) are true and the conclusion untrue. Since, according to the paracomplete theory, there's no case in which both A and $\neg A$ are true, we cannot have a counterexample to explosion. Hence, the argument form (viz., explosion) winds up being valid according to the given paracomplete theory. (*Note*. In the next chapter, we explore a logical theory according to which explosion is invalid.)

- Addition: $A \vdash_{K3} A \vee B$
  <u>*Answer*</u>. The truth conditions for disjunction tell us that $A \vee B$ is true-in-a-case just if at least one disjunct is true-in-that-case. Hence, there can't be a case c such that $c \models_1 A$ but $c \not\models_1 A \vee B$. So, the argument form is valid, according to the paracomplete theory.

- Adjunction: $A, B \vdash_{K3} A \wedge B$
  <u>*Answer*</u>. The truth conditions for conjunction tell us that $A \wedge B$ is true-in-a-case just if both conjuncts are true-in-that-case. Hence, there can't be a case c such that each of A and B is true-in-c but $A \wedge B$ is not true-in-c. So, the argument form is valid, according to the paracomplete theory.

- Simplification: $A \wedge B \vdash_{K3} A$
  <u>*Answer*</u>. The truth conditions for conjunction tell us that $A \wedge B$ is true-in-a-case just if both conjuncts are true-in-that-case. So, any case in which $A \wedge B$ is true is one in which A is true, given the running truth conditions.

- De Morgan: $\neg(A \vee B) \dashv\vdash_{K3} \neg A \wedge \neg B$
  <u>*Answer*</u> *[LRD]*. Suppose that $v(\neg(A \vee B)) = 1$, in which case $v(A \vee B) = 0$, in which case $v(A) = 0 = v(B)$, and so $v(\neg A) = 1 = v(\neg B)$, and so $v(\neg A \wedge \neg B) = 1$. (The truth/falsity conditions invoked are negation, disjunction, negation, and then conjunction, in that order.)
  <u>*Answer*</u> *[RLD]*. Conversely, suppose that $v(\neg A \wedge \neg B) = 1$, in which case $v(\neg A) = 1 = v(\neg B)$, in which case $v(A) = 0 = v(B)$, and so $v(A \vee B) = 0$ by the truth (falsity) conditions for disjunction. But, now, by the truth conditions for negation, $v(\neg(A \vee B)) = 1$.

- De Morgan: $\neg(A \wedge B) \dashv\vdash_{K3} \neg A \vee \neg B$

Answer [LRD]. Suppose that $v(\neg(A \wedge B) = 1$, in which case $v(A \wedge B) = 0$. So, by falsity conditions for conjunctions, one of A and B is false-in-v, in which case one of $\neg A$ and $\neg B$ is true-in-v, and so, by disjunction conditions, we get that $v(\neg A \vee \neg B) = 1$.
Answer [RLD]. Suppose that $v(\neg A \vee \neg B) = 1$, in which case either $v(\neg A) = 1$ or $v(\neg B) = 1$, and so either $v(A) = 0$ or $v(B) = 0$. Either way, at least one conjunct in $A \wedge B$ is false-in-v, and so $v(A \wedge B) = 0$, and hence $v(\neg(A \wedge B)) = 1$.

- Double Negation: $\neg\neg A \dashv\vdash_{K3} A$
  Answer. We can do both directions in one go by relying on the truth (falsity) conditions for negation. In particular, $v(\neg\neg A) = 1$ iff $v(\neg A) = 0$ iff $v(A) = 1$. Hence, if you've got a case v in which $\neg\neg A$ is 1, then that's a case in which A is 1 too, and vice versa.

7. *Weak Kleene.* An alternative paracomplete theory, one that is less classical than the one in this chapter, is so-called Weak Kleene (WK). On this approach, cases are as in our given paracomplete theory; however, the truth- and falsity-in-a-case conditions differ quite a bit with respect to the 'gappy' value n. In particular, the truth and falsity conditions are the same as our given paracomplete theory with respect to the *classical values* (i.e., 1 and 0); however, the conditions concerning n are as follows. If, for some WK case v, we have it that either $v(A) = \mathsf{n}$ or $v(B) = \mathsf{n}$, then $v(\neg A) = \mathsf{n} = v(\neg B)$, and similarly $v(A \vee B) = \mathsf{n} = v(A \wedge B)$. On this approach, any whiff of 'unsettledness' in a (molecular) sentence renders the entire sentence gappy (or unsettled). The question: what, if any, of the argument forms in §5.7 are valid on the WK logical theory? (Consequence, for the WK theory, is defined as usual, where *truth in a case* and *falsity in a case* are defined as per our given paracomplete theory in terms of 1 and 0, respectively.)
   Answer. First, we **note** an infelicity in the way that this problem is presented. We should clarify that negation is treated in WK exactly as in K3 (Strong Kleene): if A is a 'gap', then *any molecular in which A occurs* is also a 'gap', according to the WK approach. With this in mind, the answer is that WK agrees with K3 on all of the argument forms in §5.7 *except* for Addition, which fails in WK. A counterexample is any WK case v such that $v(A) = 1$ but $v(B) = \mathsf{n}$, in which case $v(A \vee B) = \mathsf{n}$. (Note that one philosophically intuitive motivation for this treatment is to think of the value n as recording the *meaningless* status for sentences.)

Sample answers

Answer 3i. The sentence $p \vee \neg p$ fits into none of our given categories: it's not logically true; it's not logically false; and it's not contingent. To establish this, we address each claim in turn.

3i.a. $p \vee \neg p$ is not logically true, as there are cases in which it is not true. In particular, let v be any (paracomplete) case in which p is gappy, that is, $v(p) = \mathsf{n}$. By the truth conditions for negation (see the table in §**??**), $v(\neg p) = \mathsf{n}$. By the truth conditions for disjunction, $v(p \vee \neg p) = \mathsf{n}$. Hence, any case in which p is gappy is one in which $p \vee \neg p$ is gappy, and so untrue.

3i.b. $p \vee \neg p$ is not logically false, as there are cases in which it is not false. In particular, see the case above in (3i.a).

3i.c. $p \vee \neg p$ is not contingent, as there is *no* paracomplete case in which it is false. To see this, note that $v(p \vee \neg p) = 0$ iff $v(p) = 0 = v(\neg p)$; but the truth conditions for negation require that $v(p) \neq v(\neg p)$.

Answer 4i. $p \vee \neg p$ is broadly contingent. There are cases in which it is true: for example, any case v such that $v(p) = 1$ or $v(p) = 0$ is one in which $v(p \vee \neg p) = 1$. (Why?) Moreover, there are cases in which $p \vee \neg p$ is not true: let v be any case such that $v(p) = \mathsf{n}$.

6

A Paraconsistent Theory

Exercises

1. Given your understanding of *designated values*, answer the following questions. What are the designated values of our basic classical theory (see Chapter 4)? What are the designated values of our basic paracomplete theory (see Chapter 5)?

   Answer. Both questions get the same answer: namely, that the classical and basic (non-paraconsistent) paracomplete theories recognize only one designated value, which was formally represented (in the 'formal picture') by 1.

2. One might, as mentioned in the text, have reason to reject indeterminacy but, in light of the Liar (or the like), nonetheless acknowledge overdeterminacy. A logical theory along these lines was first advanced by Asenjo (1966) but known widely from Graham Priest's work (1979) as LP for 'logic of paradox'. The difference between LP and our basic paraconsistent theory is that the former ignores incomplete cases. In particular, everything is the same except that $\mathcal{V} = \{1, \mathsf{b}, 0\}$, but $\mathcal{D}$ (the designated values) remains $\{1, \mathsf{b}\}$, as we have it. Question: are there any logical truths in LP? If so, prove it. If not, explain why not.

   Answer. Yes, there are logical truths in LP. For example, unlike in K3 and our broader (paracomplete and paraconsistent) FDE, we excluded middle holds in LP, that is, $\vdash_{LP} A \vee \neg A$. Proof: our only semantic values are in $\{1, \mathsf{b}, 0\}$. If $v(A) \in \{1, \mathsf{b}\}$ (that is, if A is designated), then $v(A \vee \neg A) \in \{1, \mathsf{b}\}$ too—and, so, true (designated). So, suppose that $v(A) = 0$. In this case, $A \vee \neg A$ again winds up true, since LP treats negation just like the classical theory when gluts (more generally, classical values) are not involved. In particular, if $v(A) = 0$, then $v(\neg A) = 1$, and so $v(A \vee \neg A) = 1$. Hence, no matter what semantic value A gets in LP, $A \vee \neg A$ winds up true. Not only is $A \vee \neg A$ logically true in LP, but so too are all sentences that are logically true according to the classical theory—that is, every classical logical truth is logically true in LP. [This is more involved to prove in a rigorous fashion, and we don't prove it here. See either the Assenjo or Priest papers cited above.]

3. Are there any cases in which $A \wedge \neg A$ is true (designated), according to our basic paraconsistent theory? If so, give an example. If not, say why not.

Answer. Yes. Any case in which A is a glut is one in which $A \wedge \neg A$ is a glut. Since being a glut is counted as a way of being true, we therefore have cases in which $A \wedge \neg A$ is true. On the 'formal picture', we can put the point thus: if there's a case v in which $v(A \wedge \neg A) \in \{1, \mathsf{b}\}$, then that's a case in which $A \wedge \neg A$ is true (or, formally, designated). There are many FDE cases like that, in particular, any case in which $v(A) = \mathsf{b}$ is such a case. After all, $v(A) = \mathsf{b}$ only if $v(\neg A) = \mathsf{b}$, and, by conditions on conjunction, if $v(A) = \mathsf{b} = v(\neg A)$, then so too $v(A \wedge \neg A) = \mathsf{b}$.

4. Are there any cases in which $\neg(A \wedge \neg A)$ is not true (not designated), according to our basic paraconsistent theory? If so, give an example. If not, say why not.

   Answer. Yes, there are cases in which $\neg(A \wedge \neg A)$, namely, any K3 (i.e., basic-paracomplete) case counts as an FDE (i.e., basic paraconsistent) case, and there are plenty of K3 cases in which $\neg(A \wedge \neg A)$ is untrue, namely, any such case v such that $v(A) = \mathsf{n}$. On the other hand, just as in K3 and the classical theory, there are no basic-paraconsistent cases in which $\neg(A \wedge \neg A)$ is 'just false' (i.e., has semantic value 0). (Why?) We should note that there are FDE cases in which $\neg(A \wedge \neg A)$ is *false*, namely, those cases that treat A as a glut—and, so, treat $A \wedge \neg A$ as a glut, and so treat the negation of $A \wedge \neg A$ as a glut. But these cases, of course, are not ones in which $\neg(A \wedge \neg A)$ is *untrue*, which is what the question was about.

5. For each of the valid forms in §6.7, give a proof that they're valid. For each of the invalid forms, give a counterexample.

   Answer. We herein give rather abbreviated proofs (though hardly as abbreviated as they could be). It may be useful to give slightly more leisurely proofs for your students.

   - Excluded Middle: $\nvdash_{FDE} A \vee \neg A$
     Answer. Counterexample: let $v(A) = \mathsf{n}$.

   - Non-Contradiction: $\nvdash_{FDE} \neg(A \wedge \neg A)$
     Answer. Counterexample: let $v(A) = \mathsf{n}$.

   - Modus Ponens: $A \rightarrow B, A \nvdash_{FDE} B$
     Answer. Counterexample: let $v(A) = \mathsf{b}$ and $v(B) \in \{\mathsf{n}, 0\}$. Since $v(A) = \mathsf{b}$, so too is $v(\neg A) = \mathsf{b}$, and so $v(\neg A \vee B) = \mathsf{b}$, which—by definition—is all that's required for $v(A \rightarrow B) = \mathsf{b}$. So, both premises are true (in virtue of being gluts), but the conclusion is untrue.

   - Modus Tollens: $A \rightarrow B, \neg B \nvdash_{FDE} \neg A$

<u>*Answer*</u>. Counterexample: let $v(A) = 1$ and $v(B) = \mathsf{b}$, in which case $v(\neg A) = 0$ and, so, $v(\neg A \vee B) = \mathsf{b}$, and so, by definition of the arrow, we have that $v(A \rightarrow B) = \mathsf{b}$. Since, by suppose, $v(B) = \mathsf{b}$, we also have that $v(\neg B) = \mathsf{b}$. Hence, we have a case in which both premises are true (designated) but the conclusion is not. Hence, the argument form is invalid in FDE. [** Additional question for students: is the argument valid in LP? (Answer: no. The same counterexample applies.)]

- Disjunctive Syllogism: $A \vee B, \neg A \nvdash_{FDE} B$
  <u>*Answer*</u>. Counterexample: let $v(A) = \mathsf{b}$ and $v(B) \in \{\mathsf{n}, 0\}$.

- Contraposition: $A \rightarrow B \dashv\vdash_{FDE} \neg B \rightarrow \neg A$
  <u>*Answer*</u>. Proof: we can do both directions at once by recalling the definition of the arrow. In particular, recall that $A \rightarrow B$ is equivalent to $\neg A \vee B$, that is, that one is true-in-a-case just if the other is true-in-that-case. Hence, $\neg B \rightarrow \neg A$ is equivalent to $\neg\neg B \vee \neg A$, which is equivalent to $B \vee \neg A$, which is equivalent to $\neg A \vee B$. So, any case in which $A \rightarrow B$ is true is one in which $\neg B \rightarrow \neg A$ is true, *and vice versa.*

- Explosion: $A, \neg A \nvdash_{FDE} B$
  <u>*Answer*</u>. Counterexample: let $v(A) = \mathsf{b}$ and $v(B) \in \{\mathsf{b}, 0\}$.

- Addition: $A \vdash_{FDE} A \vee B$
  <u>*Answer*</u>. Proof: let $v(A) \in \{1, \mathsf{b}\}$, in which case, via disjunction conditions, $v(A \vee B) \in \{1, \mathsf{b}\}$, regardless of B's value.

- Adjunction: $A, B \vdash_{FDE} A \wedge B$
  <u>*Answer*</u>. Proof: suppose that $v(A)$ and $v(B)$ are true (i.e., designated). As an inspection of the 'truth tables' for conjunction reveals, $A \wedge B$ is likewise designated. (Another way to see this is to recall that, according to FDE, a conjunction is true iff both conjuncts are true. Hence, any case in which A and B are true—be they 'just true' or at least one of them a glut—is a case in which their conjunction is true.)

- Simplification: $A \wedge B \vdash_{FDE} A$
  <u>*Answer*</u>. Proof: the clauses on conjunction tell us that $A \wedge B$ is true-in-a-case if and only if both A and B are true-in-that-case. Hence, there's no case in which $A \wedge B$ is true but B untrue.

- De Morgan: $\neg(A \vee B) \dashv\vdash_{FDE} \neg A \wedge \neg B$
  <u>*Answer*</u> *[LRD]*. We have to check both options for 'being true' (designated). Suppose, first, that $v(\neg(A \vee B)) = 1$, in which case $v(A \vee B) = 0$, and so $v(A) = 0 = v(B)$, and so $v(\neg A) = 1 = v(\neg B)$. Hence, if the

premise is 'just true', then the conclusion is true. Suppose, now, that $v(\neg(A \vee B)) = \mathsf{b}$, in which case $v(A \vee B) = \mathsf{b}$, in which case at least one of $v(A)$ and $v(B)$ is b, and the other is in $\{\mathsf{b}, 0\}$. (This fact is evident in the tables for disjunction.) But, then, both $v(B)$ and $v(A)$ are in $\{\mathsf{b}, 0\}$, and so—as reflection on negation conditions reveals—we have that $v(\neg B)$ and $v(\neg A)$ are in $\{1, \mathsf{b}\}$, and hence $v(\neg A \wedge \neg B) \in \{1, \mathsf{b}\}$, that is, $\neg A \wedge \neg B$ is designated in v. Hence, if the premise is a glut (true and false), then the conclusion is at least true. And that's all that matters for FDE validity.
<u>*Answer*</u> *[RLD]*. suppose that $v(\neg A \wedge \neg B) = 1$, in which case $v(\neg A) = 1 = v(\neg B)$, and so $v(A) = 0 = v(B)$, and so $v(A \vee B) = 0$, and so $v(\neg(A \vee B)) = 1$. Suppose, in turn, that $v(\neg A \wedge \neg B) = \mathsf{b}$, in which case both $v(A)$ and $v(B)$ are in $\{1, \mathsf{b}\}$, and at least one of $v(A)$ and $v(B)$ is b. Since both $v(A)$ and $v(B)$ are designated, so too is $v(A \vee B)$, that is, $v(A \vee B) \in \{1, \mathsf{b}\}$. But since at least one of $v(A)$ and $v(B)$ is b, we have—by conjunction conditions—that $v(A \wedge B)$ is b. Hence, for any case v, if the premise is at least true, so too is the conclusion.

- De Morgan: $\neg(A \wedge B) \dashv\vdash_{FDE} \neg A \vee \neg B$
<u>*Answer*</u>. Proof: we consider each option for designation, and do both directions for each option. To begin, $v(\neg(A \wedge B) = 1$ iff $v(A \wedge B) = 0$ iff either $v(A) = 0$ or $v(B) = 0$, iff either $v(\neg A) = 1$ or $v(\neg B) = 1$, iff $v(\neg A \vee \neg B) = 1$. This suffices for the option in which the premise is 'just true' (i.e., represented as having value 1). In turn, $v(\neg(A \wedge B)) = \mathsf{b}$ iff $v(A \wedge B) = \mathsf{b}$ iff both $v(A)$ and $v(B)$ are in $\{1, \mathsf{b}\}$ and one of them is b, iff at least one of $v(\neg A)$ and $v(\neg B)$ is b, iff $v(\neg A \vee \neg B)$ is designated (i.e., at least a glut). [The tables for these operators makes this clear on reflection.]

- Double Negation: $\neg\neg A \dashv\vdash_{FDE} A$
<u>*Answer*</u>. Proof: we can do this in one go by recalling negation conditions in FDE. In particular, $v(\neg\neg A) \in \{1, \mathsf{b}\}$ iff $v(\neg A) \in \{\mathsf{b}, 0\}$ iff $v(A) \in \{1, \mathsf{b}\}$. So, any case in which $\neg\neg A$ is at least true (i.e., designated) is one in which A is too, *and vice versa*. [It might be useful to recall that negation conditions are standard: a negation is at least true iff its negatum is at least false, and a negation is at least false iff its negatum is at least true. Here, 'at least true' means *either true only or true and false*. Formally, *at least true* amounts to *designated*, while *at least false*, in FDE, amounts to $\{\mathsf{b}, 0\}$.]

6. Explain why the following claim is true: if $\mathcal{X} \vdash_{FDE} A$ then $\mathcal{X} \vdash_{K3} A$.
<u>*Answer*</u>. This question is sufficiently answered in the Sample Answers section of this chapter.

7. Are the following argument forms valid in our basic paraconsistent (and paracomplete) theory (viz., FDE)? Provide a proof (of validity) or counterexample (for invalidity) in each case. Also, note whether or not the given forms are valid in the basic paracomplete (viz., K3) or basic classical theories (see previous chapters).
   (a) $A \to B, \neg A \to B \therefore B$
   Answer. Focus on the primitive forms of the premises, namely, $\neg A \vee B$ and $\neg\neg A \vee B$.
      - Consider, first, the classical theory. Classically, we cannot have both $\neg A$ and $\neg\neg A$ be designated. Hence, unless B is designated, not both of $\neg A \vee B$ and $\neg\neg A \vee B$ can be true (designated). Hence, there's no *classical* case in which both premises are true without B's being true too. So, according to the classical theory, the argument form is valid.
      - Now, the basic paracomplete theory (viz., K3) agrees with the classical theory in having only one designated value, and agrees with the classical theory that not both $\neg A$ and $\neg\neg A$ can be designated. Hence, the argument above goes through for K3 too: the argument form is valid according to K3. [*Note*. It's worth students pausing to reflect briefly on the paracomplete case. Intuitively, reading the arrow as 'if', we should expect the given argument form to be *invalid* for paracomplete logics, since its validity would seem to presuppose that either A or $\neg A$ is true—that is, presuppose something akin to excluded middle. That the argument forms is valid in K3 might be taken to indicate that K3 lacks a connective that accurately models the behavior of 'if' (in English, say). Of course, this point may well be used against all of the logics canvassed so far—or in the book on the whole!]
      - What about our target basic paraconsistent theory FDE? Here, the argument form is *invalid*. Here, we *can* have both $\neg A$ and $\neg\neg A$ designated: just let $v(A) = \mathsf{b}$. Now, for a counterexample to the argument form, we simply let $v(A) = \mathsf{b}$ and $v(B) \in \{\mathsf{n}, 0\}$.

   (b) $(A \vee B) \wedge C,\ A \to \neg C \therefore B$
   Answer. Again, focus on the primitive form of the second premise, namely, $\neg A \vee \neg C$.
      - Classically, the argument form is valid. Classically, we have only one way to make the conclusion untrue, namely, $v(B) = 0$. Moreover, we have only one way of make the premises true: both must have value 1, and in particular $v((A \vee B) \wedge C) = 1$, which requires that $v(C) = 1$. Now, since $v(B) = 0$, we can have $v(A \vee B) = 1$ only if $v(A) = 1$. But, now, look at the second premise (here displayed in primitive form): $\neg A \vee \neg C$. We have it so far that

$v(C) = 1$ and $v(A) = 1$, and so $v(\neg A) = 0 = v(\neg C)$, and so $v(\neg A \vee \neg C) = 0$. Hence, there's no way to make the conclusion of the argument form untrue without having at least one of the premises untrue. And this means that we can't have a case in which all premises are true but conclusion untrue.

- The argument above concerning classical logic goes through for K3; the only difference is that K3 acknowledges more than one way to be untrue. To see that this makes no difference to the validity of the given argument form, change the above proof (concerning the classical theory) at all points where B is mentioned, and flip $v(B) = 0$ to $v(B) = \mathsf{n}$. All steps go through again.
- What of FDE? Here, the argument form is *invalid*. Not surprisingly, this turns on the fact that we can designate both premises by letting A be a glut. Specifically, in the 'formal picture', we let $v(A) = \mathsf{b}$ and $v(B) \in \{\mathsf{n}, 0\}$ and $v(C) = 1$. (NB: you could also let $v(C) = \mathsf{b}$, but we choose to be concrete here and select just one value—namely, the classical one.) In this case, we have $v(A \vee B) = \mathsf{b}$, and similarly $v((A \vee B) \wedge C) = \mathsf{b}$, and similarly $v(A \rightarrow \neg C) = \mathsf{b}$. So, all premises are at least true (i.e., designated), but the conclusion isn't.

(c) $A \rightarrow B, B \rightarrow C \therefore A \rightarrow C$
<u>*Answer*</u>. Again, focus on the primitive forms of the premises, namely, $\neg A \vee B$ and $\neg B \vee C$, and similarly conclusion $\neg A \vee C$.

- Classically, this is valid. Suppose, for reductio, that we have some classical case v such that $v(A \rightarrow C) = 0$ but $v(A \rightarrow B) = 1 = v(B \rightarrow C)$. Then, since $v(A \rightarrow C) = 0$, we have that $v(A) = 1$ but $v(C) = 0$. Now, either $v(B) = 1$ or $v(B) = 0$. If the former, then $v(B \rightarrow C) = 0$, which contradicts that $v(B \rightarrow C) = 1$. If the latter, then $v(A \rightarrow B) = 0$, which contradicts that $v(A \rightarrow B) = 1$. Either way, we have a contradiction. Hence, we conclude that there can't be a classical case in which both premises are true but conclusion untrue.
- What of K3? The reasoning above, concerning the classical case, goes through here too. The only other option to consider here is the case in which the conclusion is *gappy* (versus false). So, suppose that $v(A \rightarrow C) = \mathsf{n}$, in which case $v(\neg A \vee C) = \mathsf{n}$, in which case at least one of $v(\neg A)$ and $v(C)$ is n and the other is in $\{\mathsf{n}, 0\}$. (See 'truth tables' for disjunction to see that last point.) But, now, $v(B) = 1$ or $v(B) \neq 1$. In the former case, $v(B \rightarrow C) \neq 1$, since $v(\neg B \vee C) \in \{\mathsf{n}, 0\}$ if both $v(B)$ and $v(C)$ are in $\{\mathsf{n}, 0\}$ (and we're supposing that they are). But this contradicts the supposition—made explicit in the reasoning above con-

cerning the classical theory—that $v(B \rightarrow C) = 1$. If, on the other hand, $v(B) \neq 1$, then $v(A \rightarrow B) \neq 1$, since $v(\neg A \vee B) \in \{\mathsf{n}, 0\}$ if, as we're supposing, both $v(\neg A)$ and $v(B)$ are in $\{\mathsf{n}, 0\}$. But this contradicts the supposition (made explicit in the previous reasoning) that $v(A \rightarrow B) = 1$. Either way, we can't have a K3 case in which $A \rightarrow B$ and $B \rightarrow C$ are true (designated) but $A \rightarrow C$ not true (not designated). So, the given argument form is valid in K3.

- What of FDE? Here, the argument form is *in*valid. A counterexample is one in which $v(B) = \mathsf{b}$ and $v(A) = 1$ and $v(C) \in \{\mathsf{n}, 0\}$. Working out the values of the molecular sentences reveals this to be a case in which all premises are true (designated) but conclusion untrue (undesignated).

Sample answers

Answer 6. The key point to see is that the FDE and K3 (i.e., our basic glutty–gappy and basic gappy) theories agree on the truth conditions for all connectives; it's just that the former theory acknowledges more 'semantic options' (notably, gluts) than the latter acknowledges. Close observation shows that, if you ignore the gluts (e.g., the value b) in the FDE (basic glutty–gappy) theory, you simply wind up with K3 (i.e., our basic gappy-but-no-gluts theory)! In other words, FDE and K3 agree on all cases that don't involve gluts: whatever the one counts as a counterexample, the other counts as a counterexample (provided that, as above, we're ignoring gluts). In yet other words: any case that K3 counts as a case (and, hence, as a potential counterexample) is one that FDE counts as a case (and, hence, as a potential counterexample). Hence, the K3 cases are a subset of the FDE cases. And that's the key insight: if there's no FDE counterexample to an argument, then there's no K3 counterexample to the argument. Hence, if $\mathcal{X} \vdash_{FDE} A$ (i.e., there's no basic glutty–gappy counterexample to an argument), then $\mathcal{X} \vdash_{K3} A$ (i.e., there's no basic gappy counterexample to the argument).

PART III

INNARDS, IDENTITY, AND QUANTIFIERS

7

Atomic Innards

Exercises

1. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1, 2, 3\}$, and $\delta(a) = 1$, $\delta(b) = 2$, and $\delta(d) = 3$, and $F^+ = \{1, 2\}$ and $F^- = \{1\}$. For each of the following, say whether it is true or false. If true, say why. If false, say why.
   (a) $c \models_1 Fa$
   Answer. This is true, since $\delta(a) \in F^+$.

   (b) $c \models_0 Fa$
   Answer. This is true, since $\delta(a) \in F^-$.

   (c) $c \models_1 \neg Fa$
   Answer. This is true, since, as in exercise (1b), we have that $c \models_0 Fa$.

   (d) $c \models_1 Fb \vee Fd$
   Answer. This is true, since $\delta(b) \in F^+$, and so $c \models_1 Fb$, and so, via disjunction conditions, $c \models_1 Fb \vee Fd$.

   (e) $c \models_1 Fb \wedge Fd$
   Answer. This is not true, since Fd is a gap-in-c, that is, neither true-in-c nor false-in-c, and so the conjunction of Fd and Fb is a gap-in-c. That Fd is a gap-in-c follows from the fact that $\delta(d) \notin F^+$ and $\delta(d) \notin F^-$.

   (f) $c \models_1 \neg(Fb \vee Fd)$
   Answer. This is not true. Note, first, that Fb is 'just-true-in-c', that is, $c \models_1 Fb$ and $c \not\models_0 Fb$. In turn, recall, from exercise (1e), that Fd is a gap-in-c. Putting these facts together, we have that $Fb \vee Fd$ is itself *just*-true-in-c, that is, $c \models_1 Fb \vee Fd$ and $c \not\models_0 Fb \vee Fd$. But, then, the negation of $Fb \vee Fd$ is just-*false*-in-c, that is, $c \models_0 \neg(Fb \vee Fd)$ but $c \not\models_1 \neg(Fb \vee Fd)$.

   (g) $c \models_1 Fd \rightarrow Fb$
   Answer. This is true. To see this, simply recall that $Fd \rightarrow Fb$ is equivalent to $\neg Fd \vee Fb$, and recall that Fb is true-in-c. Hence, regardless of the semantic status of $\neg Fd$ (which status happens to be *gappy*-in-c), the disjunction of Fb and $\neg Fd$ is true-in-c.

2. Construct a case in which $Fa \wedge \neg Fb$ is true.[1]
   Answer. Let $c = \langle D, \delta \rangle$ where $D = \{1, 2\}$ and $\delta(a) = 1$ and $\delta(b) = 2$ and $F^{+}\{1\}$ and $F^{-} = \{2\}$. [NB: we could have other cases that make the given sentence true. We could, e.g., have a one-element domain $\{1\}$ with $\delta(a) = \delta(b)$ and $F^{+} = F^{-} = \{1\}$, but we here give classical models when available.]

3. Construct a case in which $Fa \wedge \neg Fa$ is true.
   Answer. Let $c = \langle D, \delta \rangle$ where $D = \{1\}$ and $\delta(a) = 1$ and $F^{+} = F^{-} = D$.

4. Construct a case in which $Fa \vee \neg Fa$ is neither true nor false.
   Answer. Let $c = \langle D, \delta \rangle$ where $D = \{1\}$ and $\delta(a) = 1$ and $F^{+} = F^{-} = \emptyset$.

5. Notice that, without imposing further constraints, a case c might let $\emptyset$ be both the extension and antiextension of any (or all!) predicate(s) Π. What does this tell you about logical truths—sentences true-in-*all cases*?
   Answer. It tells us that there are no logical truths in FDE. In particular, consider the model in which $\emptyset$ is the extension and antiextension of all predicates. This model is one in which all sentences are gappy. [Proving this requires a bit of a work, and we skip it here. But your students should be able to at least roughly see why the result holds.]

Sample answers

Answer 1c. $c \models_1 \neg Fa$ iff $c \models_0 Fa$ iff $\delta(a) \in F^{-}$. Since our given case c is such that $\delta(a) \in F^{-}$ (since $\delta(a)$ is 1, which is in the antiextension of F in our given case), we conclude that $c \models_1 \neg Fa$.

Answer 2. A case in which $Fa \wedge \neg Fb$ is true as follows. Let $c = \langle D, \delta \rangle$, where $D = \{1, 2\}$ and $\delta(a) = 1$, $\delta(b) = 2$, and $F^{+} = \{1\}$ and $F^{-} = \{2\}$. (NB: there are many other cases in which $Fa \wedge \neg Fb$ is true.)

[1] To construct a case, you have to specify the domain, the denotations of the various names, and the extensions and antiextensions of given predicates.

8

Identity

Exercises

1. What, in your own words, is the difference between a logical and non-logical expression? Why is the distinction important for specifying a logical theory (a theory's account of validity)?
   Answer. We leave this to the student.

2. Discuss the following argument: not everything is identical to itself. After all, I weighed less than 10 pounds when I was born, and I weigh much more than that today. If I were identical to myself, then I'd both weigh less than 10 pounds and more than 10 pounds, but this is impossible.
   Answer. There are lots of different things one might say about this argument, and we leave it to the students to think of things. (The same goes for the first question above.) One idea we'd like to flag is the thought that perhaps identity statements—and, indeed, statements in general—are evaluated *with respect to times.* In particular, instead of having statements assessed merely *in a case*, we might have that they're assessed *at a time in a case.* Making sense of this sort of idea is one natural step towards logical theorizing (and, indeed, points towards one branch of philosophical logic called *tense logic* or *temporal logic.*

3. The following questions concern the broad logical theory as sketched in this (and the previous) chapter.
   (a) Can there be any case c in which $b = b$ is untrue (i.e., $c \not\models_1 b = b$)? If so, specify such a case.
   Answer. No. One constraint on models is that the extension of the identity predicate contains all 'identity pairs' formed out of the domain (i.e., all pairs $\langle x, x\rangle$ for all $x \in D$).

   (b) Can there be any case c in which $b = b$ is false (i.e., $c \models_0 b = b$)? If so, specify such a case.
   Answer. Yes. Just have a model such that $\langle \delta(b), \delta(b)\rangle \in \mathcal{E}^-_=$.

   (c) Can there be cases in which $b = c$ is neither true nor false?
   Answer. Yes. Just have a model such that $\langle \delta(b), \delta(c)\rangle \notin \mathcal{E}^+_=$ and $\langle \delta(b), \delta(c)\rangle \notin \mathcal{E}^-_=$, where, of course, $\delta(b) \neq \delta(c)$. (See answer to exercise (3a) for why this non-identity has to hold if $\langle \delta(b), \delta(c)\rangle$ is not

in the extension of the identity predicate.)

(d) Can there be cases in which $b = c$ is both true and false?
<u>*Answer*</u>. Yes. Just have a model such that $\langle \delta(b), \delta(c) \rangle \in \mathcal{E}^+_=$ and $\langle \delta(b), \delta(c) \rangle \in \mathcal{E}^-_=$.

4. What constraints would you impose on cases (particularly, the antiextension of identity) to rule out 'gappy' identity claims (where an identity claim is gappy in a case iff neither it nor its negation is true-in-the-given-case)?
<u>*Answer*</u>. This is sufficiently answered in Sample Answers.

5. What constraints would you impose on cases to rule out 'glutty' identity claims (where an identity claim is glutty in a case iff both it and its negation are true-in-the-given-case)?
<u>*Answer*</u>. A sufficient condition is that $\mathcal{E}^+_= \cap \mathcal{E}^-_= = \emptyset$ for all models.

6. What constraints would you impose on cases to ensure that (as per classical thinking) every identity sentence is either true or false but not both (i.e., true-in-a-case or false-in-a-case, but not both)?
<u>*Answer*</u>. We impose the previous two conditions, namely, $\mathcal{E}^+_= \cup \mathcal{E}^-_= = D \times D$ and $\mathcal{E}^+_= \cap \mathcal{E}^-_= = \emptyset$.

Sample answers

Answer 4. First, notice that some identity claims can never be gappy since they are true-in-*all cases*: namely, all of those identity claims of the form $\alpha = \alpha$. (A glance at the constraints on the extension of '$=$' shows that $\alpha = \alpha$ is true-in-*all of our cases*, for any name α.) On the other hand, we can get gappy identity claims that involve more than one name (e.g., $a = b$ or the like). (See your answer to Exercise 3.c above.) To remove such gaps from identity claims, we simply demand that, for every case c, the union of the extension and antiextension of identity (i.e., of '$=$') contains all ordered pairs from the c's domain; in other words, we impose $\mathcal{E}^+_= \cup \mathcal{E}^-_= = D \times D$. (If you've forgotten your set-theoretic notions, you should turn back to Chapter 3 for a refresher!) With this constraint on the identity predicate, there can be no pair $\langle x, y \rangle$ of objects, with x and y from D, that's in neither the extension nor antiextension of the identity predicate. And this, given the definition of *truth in a case* and *falsity in a case*, ensures that identity claims cannot be gappy.

9

Everything and Something

Exercises

1. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1,\ 2,\ 3\}$ and $\delta(a) = 1$, $\delta(b) = 2$, and $\delta(d) = 3$, and $F^+ = \{2, 3\}$ and $F^- = \{1\}$. Additionally, where R is a binary predicate, let $R^+ = \{\langle 1, 2\rangle, \langle 1, 1\rangle\}$ and $R^- = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\}$. For each of the following, say whether it is true or false. If true, say why. If false, say why.
   (a) $c \models_1 \forall x Fx$
   (b) $c \models_0 \forall x Fx$
   (c) $c \models_1 \exists x Fx$
   (d) $c \models_0 \exists x Fx$
   (e) $c \models_1 \forall x Rxb$
   Answer. The truth conditions for the universal quantifier tell us that $c \models_1 \forall x Rxb$ iff all of Rab, Rbb, and Rdb are true-in-c. However, $c \not\models_1 Rbb$, since $\delta(b) = 2$ but $\langle 2, 2\rangle \notin R^+$. (Similarly, $c \not\models_1 Rdb$.) So, claim (1e) is false: $c \not\models_1 \forall x Rxb$.

   (f) $c \models_0 \forall x Rxb$
   Answer. The falsity conditions for the universal quantifier tell us that $c \models_0 \forall x Rxb$ iff at least one of Rab, Rbb, or Rdb is false-in-c. In fact, $c \models_0 Rab$, since $\delta(a) = 1$, $\delta(b) = 2$, and $\langle 1, 2\rangle \in R^-$. So, claim (1f) is true: $c \models_0 \forall x Rxb$.

   (g) $c \models_1 \exists x Rax$
   Answer. We know that $c \models_1 \exists x Rax$ iff at least one of Raa, Rab, or Rad is true-in-c. And indeed, Raa is true-in-c, since $\delta(a) = 1$ and $\langle 1, 1\rangle \in R^+$. (Similarly, Rab is true-in-c.) So, claim (1g) is true: $c \models_1 \exists x Rax$.

   (h) $c \models_0 \exists x Rax$
   Answer. We know that $c \models_0 \exists x Rax$ iff all of Raa, Rab, and Rad are false-in-c. And in fact, this is the case, since $\langle 1, 1\rangle, \langle 1, 2\rangle$, and $\langle 1, 3\rangle$ are all in R^-. So, claim (1h) is true: $c \models_0 \exists x Rax$.

   (i) $c \models_1 \forall x(Rab \to Fx)$
   Answer. We know that $c \models_1 \forall x(Rab \to Fx)$ iff all of $Rab \to Fa$, $Rab \to Fb$, and $Rab \to Fd$ are true-in-c. Remember, a sentence of

the form $A \to B$ is true-in-c iff either A is false-in-c or B is true-in-c. Here, A is Rab, in all three of the sentences we're concerned with. Since $\delta(a) = 1$, $\delta(b) = 2$, and $\langle 1, 2\rangle \in R^-$, Rab is false-in-c. Thus, $Rab \to B$ is true-in-c, no matter what B is; all three of the sentences we're concerned with are thus true-in-c. So, claim (1i) is true: $c \models_1 \forall x(Rab \to Fx)$.

(j) $c \models_1 \exists x \forall y Rxy$
Answer. We know that $c \models_1 \exists x \forall y Rxy$ iff at least one of $\forall y Ray$, $\forall y Rby$, or $\forall y Rdy$ is true-in-c. But none of these is true-in-c. Let's focus on $\forall y Ray$. This is true-in-c iff all of Raa, Rab, and Rad are true-in-c, but since $\delta(a) = 1$, $\delta(d) = 3$, and $\langle 1, 3\rangle \notin R^+$, Rad is not true-in-c. Thus, $\forall y Ray$ isn't true-in-c either. Similar reasoning will show that neither $\forall y Rby$ nor $\forall y Rdy$ is true-in-c. Since none of the three sentences we're concerned with is true-in-c, neither is $\exists x \forall y Rxy$. So, claim (1j) is false: $c \not\models_1 \exists x \forall y Rxy$.

(k) $c \models_1 \neg\exists x \forall y Rxy$
Answer. We know that $c \models_1 \neg\exists x \forall y Rxy$ iff $c \models_0 \exists x \forall y Rxy$, and that this latter holds iff all of $\forall y Ray$, $\forall y Rby$, and $\forall y Rdy$ are false-in-c. Consider $\forall y Rby$. This is false-in-c iff at least one of Rba, Rbb, or Rbd is false-in-c, but none of these is false-in-c. After all, none of $\langle 2, 1\rangle$, $\langle 2, 2\rangle$, or $\langle 2, 3\rangle$ appears in R^-. Thus, $\forall y Rby$ is not false-in-c. (Similar reasoning can be used to show that $\forall y Rdy$ is also not false-in-c.) Thus, $c \not\models_0 \exists x \forall y Rxy$, and so claim (1k) is false: $c \not\models_1 \neg\exists x \forall y Rxy$.

(l) $c \models_0 \forall x \exists y Rxy$

2. Construct a case in which $\forall x Gx \vee \forall x \neg Gx$ is not true.[1]
Answer. There are many such cases. Here is one: a case $c = \langle D, \delta\rangle$ such that $D = \{1, 2\}$, $\delta(a) = 1$, $\delta(b) = 2$ (if there are other names around, δ can do whatever you like with them), $G^+ = \{1\}$, and $G^- = \{1\}$ (if there are other predicates around, they can receive any old extensions and antiextensions). Given this setup, we have that $c \not\models_1 Gb$, so $c \not\models_1 \forall x Gx$. We also have that $c \not\models_0 Gb$, so $c \not\models_1 \neg Gb$, and so $c \not\models_1 \forall x \neg Gx$. Putting these together, we have that $c \not\models_1 \forall x Gx \vee \forall x \neg Gx$. (Note that a played no role in the reasoning here; the reason it's important that $\delta(a) = 1$ is so we are sure that everything in the domain of c has a name in c.)

3. Give a case in which $\forall x(Gx \to Hx)$ is true but $\forall x(Gx \wedge Hx)$ is not true.
Answer. There are many such cases. Here is one: a case $c = \langle D, \delta\rangle$ such that $D = \{23\}$, $\delta\alpha = 23$ for every name α, $G^+ = \emptyset$, $G^- = \{23\}$, $H^+ = \{23\}$, and $H^- = \emptyset$. Since there is only one member of the domain, it doesn't

[1] As in Chapter 7, to construct a case, you have to specify the domain, the denotations of the various names, and the extension and antiextensions of given predicates.

matter which name we use; what goes for one name will go for all the others. Let's take a. We have that $c \models_1 Ga \rightarrow Ha$, since $c \models_1 \neg Ga$. (We also have that $c \models_1 Ha$, which would suffice as well.) Since this will hold for every name, we have that $c \models_1 \forall x(Gx \rightarrow Hx)$. On the other hand, we have that $c \not\models_1 Ga$, so $c \not\models_1 Ga \wedge Ha$. Thus, $c \not\models_1 \forall x(Gx \wedge Hx)$.

4. The argument from §9.1 about Agnes and cats, from (3) and (4) to (5), has the form $\forall x(Cx \rightarrow Sx), Ca \therefore Sa$. In which of our three logical theories is this argument valid? (Give a counterexample for any theory in which the argument is invalid.)

   Answer. This argument is valid as long as the Exclusion constraint is in place; thus, it is valid according to the paracomplete (non-paraconsistent) account, and according to the classical account. There are many (paraconsistent) counterexamples. Here is one: a case $c = \langle D, \delta \rangle$ such that $D = \{\text{Shakespeare}\}$, $\delta(\alpha) = \text{Shakespeare}$ for every name α, $C^+ = \{\text{Shakespeare}\}$, $C^- = \{\text{Shakespeare}\}$, $S^+ = \emptyset$, and $S^- = \{\text{Shakespeare}\}$. For any name α, we have that $c \models_0 C\alpha$, and thus $c \models_1 \neg C\alpha$. Therefore, $c \models_1 C\alpha \rightarrow S\alpha$. Since this holds for every name α, $c \models_1 \forall x(Cx \rightarrow Sx)$. The first premise is thus true-in-c. As for the second premise, we can see directly that $c \models_1 Ca$, since $\delta(a) \in C^+$. But $c \not\models_1 Sa$, since $\delta(a) \notin S^+$. Thus, c is a counterexample to this argument. (Note that c doesn't satisfy the Exclusion constraint.)

5. Which of the following best expresses that *nothing is horrible*? Justify your answer by appealing to the truth and falsity conditions of the quantifiers.

   (a) $\neg\forall x Hx$
   (b) $\neg\exists x Hx$

   Answer. The best choice is (b): $\neg\exists x Hx$. The claim that *nothing is horrible* is true iff everything is *not horrible*; equivalently, iff everything is in the antiextension of *horrible*. And indeed, $\neg\exists x Hx$ is true-in-a-case-c iff $\exists x Hx$ is false-in-c, which is the case iff $H\alpha$ is false-in-c for every name α. If H represents *horrible*, then these are just the truth conditions we're after. On the other hand, $\neg\forall x Hx$ is true-in-a-case-c iff $\forall x Hx$ is false-in-c, which is the case iff $H\alpha$ is false-in-c for *some particular* name α. If H represents *horrible*, then these are not the truth conditions we want: these are the truth conditions of *There is something that's not horrible*, or *Not everything is horrible*.

   Now, let's turn to falsity conditions. The claim that *nothing is horrible* is false iff something *is* horrible; equivalently, if something is in the extension of *horrible*. Indeed, $\neg\exists x Hx$ is false-in-a-case-c iff $\exists x Hx$ is true-in-c, which is the case iff $H\alpha$ is true-in-c for some particular name α. If H represents *horrible*, these are just the falsity conditions we're after. On the other hand, $\neg\forall x Hx$ is false-in-a-case-c iff $\forall x Hx$ is true-in-c, which is the case iff $H\alpha$ is true for every name α. If H represents *horrible*, then these are not the falsity conditions we want: these are the falsity conditions of, again, *There is something that's not horrible*, or *Not everything is horrible*.

6. Do the two-way-validity claims hold in our current paraconsistent (and paracomplete) theory with quantifiers—a theory we'll call 'FDE' (even though we now have quantifiers)? Justify your answer.
   (a) $\forall xFx \dashv\vdash \neg\exists x\neg Fx$
   Answer. Yes, this two-way validity claim holds. To show this, we need to show two things: first, that any case c such that $c \models_1 \forall xFx$ is also such that $c \models_1 \neg\exists x\neg Fx$; and second, that any case c such that $c \models_1 \neg\exists x\neg Fx$ is also such that $c \models_1 \forall xFx$.

   The first part first. Suppose $c = \langle D, \delta\rangle$ is such that $c \models_1 \forall xFx$. Then every name α must be such that $c \models_1 F\alpha$. This means that every name α must be such that $c \models_0 \neg F\alpha$, which is enough to guarantee that $c \models_0 \exists x\neg Fx$, and so $c \models_1 \neg\exists x\neg Fx$.

   The second part second. Suppose $c = \langle D, \delta\rangle$ is such that $c \models_1 \neg\exists x\neg Fx$. Then $c \models_0 \exists x\neg Fx$, and so every name α must be such that $c \models_0 \neg F\alpha$. Thus, every name α is such that $c \models_1 F\alpha$, and so $c \models_1 \forall xFx$.

   (b) $\exists xFx \dashv\vdash \neg\forall x\neg Fx$
   Answer. Yes, this two-way validity claim holds. To show this, we need to show two things: first, that any case c such that $c \models_1 \exists xFx$ is also such that $c \models_1 \neg\forall x\neg Fx$; and second, that any case c such that $c \models_1 \neg\forall x\neg Fx$ is also such that $c \models_1 \exists xFx$.

   The first part first. Suppose $c = \langle D, \delta\rangle$ is such that $c \models_1 \exists xFx$. Then there must be some name α such that $c \models_1 F\alpha$. This means that α must be such that $c \models_0 \neg F\alpha$, which is enough to guarantee that $c \models_0 \forall x\neg Fx$, and so $c \models_1 \neg\forall x\neg Fx$.

   The second part second. Suppose $c = \langle D, \delta\rangle$ is such that $c \models_1 \neg\forall x\neg Fx$. Then $c \models_0 \forall x\neg Fx$, and so some name α must be such that $c \models_0 \neg F\alpha$. Thus, α is such that $c \models_1 F\alpha$, and so $c \models_1 \exists xFx$.

Sample answers

Answer 1a. Claim (1a) is false: $\forall xFx$ is not true-in-our given case. To show as much, we invoke the truth conditions for the universal quantifier, which has it that $c \models_1 \forall xFx$ iff $c \models_1 F\alpha$ for all names α such that $\delta(\alpha) \in D$. Hence, if each of Fa, Fb, and Fd are true-in-our given c, then so too is $\forall xFx$. To figure out whether these (atomic) sentences are true, we need to consult the truth conditions for atomics. Quick consultation reveals that Fa is true in c iff $\delta(a)$ is in F^+. But $\delta(a) = 1$, and $1 \notin F^+$. Hence, $c \not\models_1 Fa$, and so, as above, $c \not\models_1 \forall xFx$.

Answer 1b. Claim (1b) is true: $\forall xFx$ is false-in-our-given-c. The falsity conditions (i.e., conditions for $\models_0$) for the universal quantifier tell us that $c \models_0 \forall xFx$ if any of Fa, Fb, or Fd are false-in-c. Figuring out whether any of these (atomic) sentences is false-in-c involves consulting the falsity conditions for atomics. Quick consultation reveals that Fa is false-in-c iff $\delta(a)$ is in F^-. But $\delta(a) = 1$, and $1 \in F^-$. Hence, $c \models_0 Fa$, and so, as above, $c \models_0 \forall x Fx$.

Answer 1c. The truth conditions for the existential quantifier tell us that $c \models_1 \exists x Fx$ iff either Fa, Fb or Fd is true-in-c. Truth conditions for these atomics reveals that $c \models_1 Fb$ (similarly for Fd) since $\delta(b) = 2$ and $2 \in F^+$ (similarly, $\delta(d) = 3$ and $3 \in F^+$). So, claim (1c) is true: $c \models_1 \exists x Fx$.

Answer 1d. The falsity conditions for the existential quantifier tell us that $c \models_0 \exists x Fx$ iff each of Fa, Fb, and Fd is false-in-c. Falsity conditions for these atomics reveals that $c \not\models_0 Fb$ (similarly for Fd) since $\delta(b) = 2$ but $2 \notin F^-$ (similarly for d with $3 \notin F^-$). So, claim (1d) is false: $c \not\models_0 \exists x Fx$.

Answer 1l. Claim (1l) is true: $\forall x \exists y Rxy$ is false-in-our-given-case. By the falsity conditions for the universal quantifier, we have that $c \models_0 \forall x \exists y Rxy$ if and only if $c \models_0 \exists y R\alpha y$ for some name α such that $\delta(\alpha) \in D$. Is there such a name α such that $c \models_0 \exists y R\alpha y$? Yes: the name a fits the bill: $c \models_0 \exists y Ray$. After all, by the falsity conditions for the existential quantifier, $c \models_0 \exists y Ray$ iff $c \models_0 Ra\alpha$ for all names α (such that $\delta(\alpha) \in D$). So, $c \models_0 \exists y Ray$ iff $c \models_0 Raa$ and $c \models_0 Rab$ and $c \models_0 Rad$. But that's exactly what we have in our given case: each of Raa, Rab and Rad is indeed false-in-c, since $\delta(a) = 1$, $\delta(b) = 2$, $\delta(d) = 3$, and the antiextension of R contains each of the pairs $\langle 1, 1 \rangle$, $\langle 1, 2 \rangle$, and $\langle 1, 3 \rangle$.

PART IV

FREEDOM, NECESSITY, AND BEYOND

10

Speaking Freely

Exercises

1. Is the following argument valid in our 'freed up' theory? Explain your answer.[1]

$$\forall x Fx \therefore Pb \to Fb$$

(Hint: don't forget about cases where $\delta(b) \notin E$!)

Answer. No, it's not. We can make a counterexample: a case c whose domain $D = \{1,2\}$, and whose set E of existent things is $\{1\}$. If we now suppose that $\delta(b) = 2$, that $F^+ = \{1\}$, and that $P^- = \emptyset$, we have a counterexample. (We can let F^- and P^+ be whatever we like.) Since all names α such that $\delta(\alpha) \in E$ are such that $c \models_1 F\alpha$, we have that $c \models_1 \forall x Fx$. However, $c \not\models_1 \neg Pb$, and $c \not\models_1 Fb$, so $c \not\models_1 Pb \to Fb$; it is a counterexample.

2. Specify a case in which $Fb \wedge Ga$, $\neg\exists x Fx$, and $\neg\exists x Gx$ are all true.

Answer. The following case will work: a case c such that $D = \{1,2,3\}$, $\delta(a) = 2$, $\delta(b) = 3$, $E = \{1\}$, $F^+ = \{3\}$, $F^- = \{1,2\}$, $G^+ = \{2\}$, and $G^- = \{1,3\}$. It's clear that $c \models_1 Fb \wedge Ga$. To see that $c \models_1 \neg\exists x Fx$, we should verify that $c \models_0 \exists x Fx$; this requires that for any name α, whenever $\delta(\alpha) \in E$, $c \models_0 F\alpha$. Since $\delta(\alpha) \in E$ iff $\delta(\alpha) = 1$, this amounts to verifying that $1 \in F^-$, which is the case. Similar reasoning shows that, since $1 \in G^-$, $c \models_1 \neg\exists x Gx$. (Many other cases will work as well.

3. Specify which of the following are valid arguments, and justify your answer.
   (a) $\forall x Fx \therefore Fa$
   (b) $Fb \wedge Gb \therefore \exists x(Fx \wedge Gx)$

   Answer. This argument is not valid on the freed-up theory. To see this, consider a case c such that no name α such that $\delta(\alpha) \in E$ is such that and $c \models_1 F\alpha \wedge G\alpha$. This will be enough to guarantee that $c \not\models_1 \exists x(Fx \wedge Gx)$. However, if $\delta(b) \notin E$, it can still be that $c \models_1 Fb \wedge Gb$; that's our counterexample.

   (c) $\neg\exists x Fx \therefore \neg Fb$

   Answer. This argument is not valid on the freed-up theory. To see this, consider a case c such that every name α such that $\delta(\alpha) \in E$ is

[1] Here (and below), we're using '$\therefore$' just to separate premises from conclusion.

such that $c \models_0 F\alpha$. This is enough to guarantee that $c \models_1 \neg\exists x Fx$. However, if $\delta(b) \notin E$, it can still be that $c \not\models_0 Fb$, and so $c \not\models_1 \neg Fb$. This is our counterexample.

(d) $\neg Fa \therefore \neg\forall x Fx$
Answer. This argument is not valid on the freed-up theory. To see this, consider a case c such that every name α such that $\delta(\alpha) \in E$ is such that $c \not\models_0 F\alpha$. This would guarantee that $c \not\models_0 \forall x Fx$, and so $c \not\models_1 \neg\forall x Fx$. However, if $\delta(a) \notin E$, it can still be that $c \models_0 Fa$, and so $c \models_1 \neg Fa$. This is our counterexample.

4. If only the objects in E exist in a given case, but $E \subseteq D$ (for any case), what is the 'ontological status' of elements in $\{x : x \in D \text{ and } x \notin E\}$, the so-called complement of E relative to D. (The complement of E relative to D is often denoted by either '$D \setminus E$' or '$D - E$')?

   Answer. There are a number of ways to answer this; one way to proceed is by example, as the chapter has. What is the ontological status of Guy Stewart, or other nonexistent things that we have names for, like Pegasus? The most natural thing to say, it seems, is that these things don't exist. That was the whole point of Guy Stewart: almost all we know about him is that he doesn't exist. But there are other options for answers that one might sensibly give here. (Perhaps Guy Stewart exists in some attenuated way, or as an abstract object, or in some other realm, or ...)

5. If 'Guy Stewart' really doesn't denote *anything*, then 'Guy Stewart' doesn't denote anything—full stop. So, if 'Guy Stewart' doesn't denote anything, then it doesn't denote anything in the 'big domain' D. What, if anything, does this suggest about our formal modeling of the matter?

   Answer. Again, a number of answers are possible. If one accepts the reasoning in the question, it seems to lead to the conclusion that the modeling method described in this chapter isn't really a sensible way to proceed in the presence of names for nonexistent things. After all, the models crucially allow that there might be some things in D but not in E—if we conclude that there can't be any such things, we are effectively sticking with our 'unfree' models.

6. You might reject that there are predicates that are true of objects that don't exist. (E.g., you might reject that 'Agnes is thinking about x' is true of the *so-called object* Guy.) Instead of drawing the lesson that some of our predicates can be true of objects that don't exist, what other lessons might you draw from the Guy Stewart story?

   Answer. There are any number of options here. Some options that are compatible with the idea that predicates cannot be true of objects that don't exist will find things that *do* exist to stand in. For example, instead of taking a name to refer (impossibly) to the nonexistent object Guy Stewart, one might take the name to refer to the existent concept of Guy Stewart. This would need to be supplemented with some kind of story about how there can be a concept of a nonexistent thing, and how concepts play into the truth of sentences like 'Guy Stewart is my imaginary friend'.

7. We have said (in this and previous chapters) that existence claims like b *exists* have the form $\exists x(x = b)$. You might be wondering about a different approach: treating 'exists' as a more standard, quantifier-free predicate. How might this go? Is the predicate to be treated as a logical expression? If so, what are the constraints on its extension and antiextension? If the predicate is non-logical (i.e., its extension and antiextension get no special constraints aside from those imposed on all predicates by the kind of cases involved), how do existence claims like b *exists* relate to existential claims like $\exists x(x = b)$? What, in general, is the logic of your proposed existence predicate? (This question is left wide open as an opportunity for you to construct your own alternative logical theory of existence.)

   Answer. As you can imagine, there is a large variety of possible answers here. One natural approach is to use the 'free' cases we've defined in this chapter, but keep the old truth- and falsity-conditions for $\forall$ and $\exists$, so that quantified sentences still depend only on D. One can then use E to interpret a new predicate $\mathbb{E}$, by defining $\mathbb{E}^+ = E$ and $\mathbb{E}^- = D - E$. This would mean that $\mathbb{E}$ is always consistent and complete. Alternately, we could go back to the old 'unfree' cases, and simply treat $\mathbb{E}$ as an ordinary predicate, allowing for existence gluts and existence gaps.

   Or, we could take either of these approaches, but stick with the 'free' clauses for the quantifiers, and use these clauses to create some connection between $\exists$ and $\mathbb{E}$. There are plenty of options!

Sample answers

Answer 3a. The argument from $\forall xFx$ to Fa is not valid (according to the current freed up theory). To see this, let c be any of our (current, freed-up) cases in which $\delta(a) \notin E$ (i.e., in which the denotation of name a is not among the objects that, according to c, exist), and let $\delta(a) \notin F^+$ (i.e., the denotation of a is not in the set of objects that, according to c, have the property F), and let everything else in D be in the extension of F (i.e., be in F^+). This case is a counterexample to the given argument.

11

Possibilities

Exercises

Unless otherwise stated, the Mfde truth conditions (see §11.4.2) are assumed in the following exercises.

1. Which of the following arguments are valid? Justify your answer (with a proof or counterexample).[1]

   (a) $\Box(Fb \wedge Fa) \therefore \Box Fb \wedge \Box Fa$

   Answer. This argument is valid. Consider any case c such that $[c, @] \models_1 \Box(Fb \wedge Fa)$. This means that, for all worlds w in c, $[c, w] \models_1 Fb \wedge Fa$. Thus, all worlds w in c must be such that $[c, w] \models_1 Fb$—and so $[c, @] \models_1 \Box Fb$—and such that $[c, w] \models_1 Fa$—and so $[c, @] \models_1 \Box Fa$. But then it must be that $[c, @] \models_1 \Box Fb \wedge \Box Fa$. So there can be no counterexample.

   (b) $\Box Fb \therefore \Diamond Fb$

   Answer. This argument is valid. Consider any case c such that $[c, @] \models_1 \Box Fb$. This means that, for all worlds w in c, $[c, w] \models_1 Fb$. In particular, it must be that $[c, @] \models_1 Fb$. But then there is a world w in c—namely @—such that $[c, w] \models_1 Fb$, and so $[c, @] \models_1 \Diamond Fb$.

   (c) $\Diamond Fb \therefore \Box Fb$

   Answer. This argument is not valid. There are many counterexamples; here is one. Consider a case $c = \langle \mathcal{W}, @, D, \mathcal{E}, \delta \rangle$ such that $\mathcal{W} = \{@, w\}$, $D = \{1\}$, $\delta(\alpha) = 1$ for every name α, $F^+_@ = \emptyset$, $F^-_@ = \{1\}$, $F^+_w = \{1\}$, and $F^-_w = \emptyset$. (It doesn't matter what $\mathcal{E}$ does here.) Since $F^+_w = \{1\}$, and $\delta(b) = 1$, we have that $[c, w] \models_1 Fb$, thus $[c, @] \models_1 \Diamond Fb$. But since $\delta(b) \notin F^+_@$, $[c, @] \not\models_1 Fb$, and so $[c, @] \not\models_1 \Box Fb$. Thus, c is a counterexample to the argument.

   (d) $\Box(a = a) \therefore \Diamond \exists x(x = a)$

   Answer. This argument is not valid. There are many counterexamples; here is one. Consider a case $c = \langle \mathcal{W}, @, D, \mathcal{E}, \delta \rangle$ such that $\mathcal{W} = \{@\}$, $D = \{1\}$, $\delta(\alpha) = 1$ for every name α, and $E_@ = \emptyset$. We have that $[c, @] \models_1 a = a$ (since the extension of $=$ must include $\langle 1, 1 \rangle$), and

[1] Here, we use '$\therefore$' merely to separate premises and conclusion. (The following are arguments from our formal language, not argument forms.)

since @ is the only world in c, this means that $[c, @] \models_1 \Box(a = a)$. However, $[c, @] \not\models_1 \exists x(x = a)$. Since there is nothing in $E_@$, no claim of the form $\exists x A$ can be true-at-@-in-c. But since @ is the only world in c, this means that $[c, @] \not\models_1 \Diamond\exists x(x = a)$. So c is a counterexample to the argument.

(e) $\Box\neg\exists x(x = a) \therefore \neg(a = a)$
<u>*Answer*</u>. This argument is not valid. There are many counterexamples; we'll build one from that counterexample offered in Answer 1d. Take the counterexample from Answer 1d, and specify in addition that the antiextension of $=$ at @ in c is empty. Since $E_@$ is empty, $[c, @] \models_0 \exists x(x = a)$. (Remember, if there is nothing in $E_@$, then *every* name α such that $\delta(\alpha) \in E_@$ (all none of them!) is such that $[c, @] \models_0 \alpha = a$.) Thus, $[c, @] \models_1 \neg\exists x(x = a)$, and since @ is the only world in c, this means that $[c, @] \models_1 \Box\neg\exists x(x = a)$. But, since the antiextension of $=$ at @ in c is empty, we know that $[c, @] \not\models_0 a = a$, so $[c, @] \not\models_1 \neg(a = a)$. Thus, c is a counterexample to the argument.

(f) $\Box\Diamond Fa \therefore \Diamond\Box Fa$
<u>*Answer*</u>. This argument is not valid. There are many counterexamples; we'll reuse the counterexample c from Answer 1c, which is a counterexample to this argument as well. Since $\delta(a) \in F^+_w$, we have that $[c, w] \models_1 Fa$, so every world w' is such that $[c, w'] \models_1 \Diamond Fa$, so $[c, @] \models_1 \Box\Diamond Fa$. On the other hand, since $\delta(a) \notin F^+_@$, we have that $[c, @] \not\models_1 Fa$, so every world w' is such that $[c, w'] \not\models_1 \Box Fa$, so $[c, @] \not\models_1 \Diamond\Box Fa$. Thus, c is a counterexample to the argument.

(g) $\neg\Diamond\exists x(x = a) \therefore \neg\Box(a = a)$
<u>*Answer*</u>. This argument is not valid. There are many counterexamples; we'll reuse the counterexample c from Answer 1e, which is a counterexample to this argument as well. We showed in Answer 1e that $[c, @] \models_0 \exists x(x = a)$. Since @ is the only world in c, this means that $[c, @] \models_0 \Diamond\exists x(x = a)$, and so $[c, @] \models_1 \neg\Diamond\exists x(x = a)$. We also showed that $[c, @] \not\models_0 a = a$, which means (since @ is the only world in c) that $[c, @] \not\models_0 \Box(a = a)$, and so $[c, @] \not\models_1 \neg\Box(a = a)$. Thus, c is a counterexample to the argument.

2. Are there cases in which $a = a$ is not true? If so, provide one. If not, show as much.

<u>*Answer*</u>. No, there are not. To be a case in which $a = a$ is not true is to be a case c such that $[c, a] \not\models_1 a = a$; that is, a case in which $\langle a, a \rangle$ is not in the extension of $=$ at @. But, given our restrictions, this can never be. Thus, there is no case in which $a = a$ fails to be true.

3. Are there cases in which $\Box(a = a)$ is not true? If so, provide one. If not, show as much.

   Answer. No, there are not. To be a case in which $\Box(a = a)$ is not true is to be a case c in which there is a world w such that $[c, w] \not\models_1 a = a$; that is, a case in which $\langle a, a\rangle$ is not in the extension of $=$ at w. But, given our restrictions, this can never be. Thus, there is no case in which $\Box(a = a)$ fails to be true.

4. Is $\Diamond(a = a)$ logically true?

   Answer. This is answered in the text (viz., 'sample answers').

5. Explain how to modify Mfde to get the following results (i.e., for the resulting, modified consequence relation). If no modification is required for a given item, prove as much.

   (a) $\Box A \wedge \Box\neg A \vdash B$

   Answer. There are a number of modifications which could work. One is the exclusion constraint. If no predicate's extension at a world overlaps with its antiextension at that world, then there can never be a world in any case at which both A and $\neg A$ hold, for any A (see answer 6a below). If $\Box A \wedge \Box\neg A$ were true in a case, though, then both A and $\neg A$ would have to hold at every point in that case. In particular, they'd have to hold at @. Since this would be ruled out by the exclusion constraint, no case satisfying the exclusion constraint could ever make $\Box A \wedge \Box\neg A$ true. Thus, no such case could provide a counterexample to this argument.

   (b) $\vdash \Box A \vee \neg\Box A$

   Answer. There are a number of modifications that could work. One is the exhaustion constraint. If every predicate's extension at a world and antiextension at that world, taken together, exhaust the domain, then $C \vee \neg C$ will always be true at every point in every case, for any sentence C (see answer 6b below). If we take the instance where C is the sentence $\Box A$, then we have the target sentence: $\Box A \vee \neg\Box A$. Since it must be true at every point in every case, it must be true at @ in every case; there will be no counterexample possible.

   (c) $\vdash \Box(A \vee \neg A)$

   Answer. There are a number of modifications that could work. Again, the exhaustion constraint will do the job. With the exhaustion constraint in place, every world in every case must make $A \vee \neg A$ true, and so @ in every case must make $\Box(A \vee \neg A)$ true.

   (d) $\Box A, \Diamond\neg A \vdash B$

<u>*Answer*</u>. There are a number of modifications that could work. One of them is the exclusion constraint. If $\Box A$ is true at @ in a case c, then A must be true at every world in c; and if $\Diamond\neg A$ is true at @ in c, then there must be some world in c where $\neg A$ is true. At that very world, both A and $\neg A$ must be true; if this is ruled out by the exclusion constraint, then no case can make both premises true, so there can be no counterexample.

(e) $\vdash \forall x \Diamond \exists y(y = x)$
<u>*Answer*</u>. This already holds in Mfde; no modification is necessary. To see this, we'll assume that there is some counterexample c, and show that the supposition goes wrong. As c a counterexample, we know that $[c, @] \not\models_1 \forall x \Diamond \exists y(y = x)$. Thus, there is some name a such that $\delta(a) \in E_@$ and $[c, @] \not\models_1 \Diamond \exists y(y = a)$. This means that there is no world w such that $[c, w] \models_1 \exists y(y = a)$. In particular, @ cannot be such a world. So $[c, @] \not\models_1 \exists y(y = a)$. From this, we can conclude that there is no name b such that $\delta(b) \in E_@$ and $\delta(b) = \delta(a)$. But there is: a itself! So there can be no such case, no counterexample.

6. Suppose that we define a world w (of a case c) to be *consistent* iff there's no sentence A such that $[c, w] \models_1 A$ and $[c, w] \models_0 A$. Similarly, suppose that we define a world w (of a case c) to be *complete* iff there's no sentence A such that $[c, w] \not\models_1 A$ and $[c, w] \not\models_0 A$ (i.e., the world is such that every sentence is either true at the world or false at the world). Precisely formulate and explore the following, narrower (though stronger) variations on Mfde.

(a) K3 Modal: the cases are all Mfde cases *that contain only consistent worlds* (though not necessarily complete).
<u>*Answer*</u>. We can get this by imposing the exclusion constraint at every world: requiring that no predicate can have an extension at a world that overlaps its antiextension at that world. Just as imposing this exclusion constraint in our non-modal systems means that we will never have a case at which both A and $\neg A$ are true, for any sentence A, so too does imposing the constraint here mean we'll never have a world in any case at which both A and $\neg A$ are true, for any sentence A.

Remember, by adding restrictions (and so removing cases), we can never make a valid argument become invalid, but we can make an invalid argument become valid. So all the arguments in exercises 1–5 that were valid in Mfde will remain valid with this new restriction. In exercise 1, no new arguments become valid as a result of this restriction; the counterexamples given above for the invalid arguments already meet this restriction. Since exercises 2–4 all present things that were valid in Mfde anyway, they will remain valid. In exercise 5, 5a and 5d would become valid upon adding this constraint, as discussed above.

(b) LP Modal: the cases are all Mfde cases *that contain only complete worlds* (though not necessarily consistent).
Answer. We can get this by imposing the exhaustion constraint at every world: requiring of every predicate that its extension at a world and its antiextension at that world, taken together, exhaust the domain. Just as imposing this exhaustion constraint in our non-modal systems means that $A \vee \neg A$ is true in every case, for any sentence A, so too does imposing the constraint here mean that every world in every case will make $A \vee \neg A$ true, for any sentence A.

As above, in exercise 1, no new arguments become valid as a result of this restriction; the counterexamples given above for the invalid arguments already meet this restriction. Since exercises 2–4 all present things that were valid in Mfde anyway, they will remain valid. In exercise 5, 5b and 5c would become valid upon adding this constraint, as discussed above.

(c) Classical (though 'free') Modal: the cases are all Mfde cases *that contain only consistent and complete worlds.*[2]
Answer. We can get this by imposing both the exclusion and the exhaustion constraint at every world. This will ensure both that every world in every case makes $A \vee \neg A$ true, for every sentence A, and that no world in any case will make both A and $\neg A$ true, for any A.

As above, in exercise 1, no new arguments become valid as a result of this restriction; the counterexamples given above for the invalid arguments already meet both restrictions. Since exercises 2–4 all present things that were valid in Mfde anyway, they will remain valid. In exercise 5, all of 5a–5d would become valid upon adding these constraints, as discussed above.

For each of the resulting logics in (6a)–(6c), go through all questions from Exercises 1–5 again but focus on the given narrower logic.

7. What other connectives might be treated along 'worlds' lines? What if, instead of thinking of the elements in $\mathcal{W}$ as *worlds*, we think of $\mathcal{W}$ as containing *points in time*. Now consider the connectives *it is always true that...* and *it is sometimes true that....* If we treat these connectives along our box and, respectively, diamond lines, what sort of 'temporal logic' (i.e., logic of such temporal connectives) do we get? Related question: what sort of 'ordering' on your points in $\mathcal{W}$ do you need to give in order to add

[2]The freedom comes from our speaking-freely approach to cases. See Chapter 10.

plausible *it will be true that...* and *it was true that...* connectives into the picture? (E.g., do your points of time have to be ordered in the way that, e.g., the natural numbers are ordered?)

Answer. This question is an invitation to think some things through; many different answers are possible. When it comes to taking an ordering into account, the answers will depend on one's picture of the structure of time (discrete? continuous? branching into the future, or one single future? branching into the past, or one single past?). There will, though, have to be some appeal to this order in the truth- and falsity-conditions for *it will be true that...* and *it was true that....* For example, one might define a connective F for *it will be true that...* as follows:

- $[c, w] \models_1 FA$ iff there is a time w' *after* w such that $[c, w'] \models_1 A$.
- $[c, w] \models_0 FA$ iff every time w' *after* w is such that $[c, w'] \models_0 A$.

Here, the clauses depend only on what happens *after* w. The truth- and falsity-conditions for *it was true that ...* should likewise depend only on what happens *before* the time in question.

Sample answers

Answer 4. Yes, $\Diamond(a = a)$ is logically true. Suppose, for reductio, that it is not logically true, in which case there must be a case (universe) in which $\Diamond(a = a)$ is untrue, that is, a case c such that $[c, @] \not\models_1 \Diamond(a = a)$. But, then, by the truth conditions for the diamond, there must be no world in our given case (universe) at which $a = a$ is true. But by the constraints on identity (viz., that the extension of the identity predicate contains *all* pairs of identical objects from the domain of the given case), $a = a$ is true at all worlds. Hence, we've run into an unacceptable contradiction (viz., that there is both some world in our case and no world in our case in which $a = a$ is true), and so we reject our initial supposition that $\Diamond(a = a)$ is not logically true. We conclude, then, that $\Diamond(a = a)$ is logically true.

12

Glimpsing Different Logical Roads

Exercises

1. Prove that $\nabla A \vdash A$ and $A \vdash \nabla A$ (i.e., that our Actuality operator, defined above, both Captures and Releases).

   <u>*Answer*</u>. Suppose that ∇A is true in in a case c; that is, that $[c, @] \models_1 \nabla A$. We know that this happens iff $[c, @] \models_1 A$, so it must be that A is also true in c. Thus, the argument from ∇A to A is valid.

   For the other direction, suppose that A is true in a case c; that is, that $[c, @] \models_1 A$. This is sufficient for ∇A to be true at every world in c. In particular, it is sufficient for ∇A to be true at @ in c. Thus, $[c, @] \models_1 \nabla A$ as well; the argument is valid.

2. Do we have that $\Box A \vdash \nabla A$ in the Mfde theory (expanded with ∇ as above)? If so, prove it. If not, give a counterexample.

   <u>*Answer*</u>. This is answered in the text (viz., 'sample answers').

3. Using the revised truth conditions (see page 166), give a countermodel to η-Explosion: viz., $A, \eta A \therefore B$.

   <u>*Answer*</u>. Consider the instance of the argument $p, \eta p \therefore q$. There are a number of counterexamples to this argument. Here's one: a case c such that $c \models_1 p$, $c \models_0 p$, and $c \not\models_1 q$. Since $c \models_0 p$, $c \models_1 \eta p$ by the revised truth conditions. (Also, since $c \models_1 p$, $c \models_0 \eta p$, but that's not important here.) Thus, both premises are true in c, but the conclusion is not; c is a counterexample.

4. With respect to the 'meaningless' approach to disjunction (see §12.4), provide a case in which A is true but $A \vee B$ not true (for some A and B).

   <u>*Answer*</u>. Consider the instance $p \therefore p \vee q$. This is counterexampled by a case c such that $c \models_1 p$, $c \not\models_1 q$, and $c \not\models_0 q$. According to the 'meaningless' approach, $c \not\models_1 p \vee q$, since $c \not\models_! q$. So c is a counterexample to this argument.

5. How does the 'meaningless' approach (see §12.4) compare with Weak Kleene (see Chapter 5 exercises)?

   <u>*Answer*</u>. The only difference is in the treatment of gluts. The Weak Kleene system discussed in the exercises of Chapter 5 imposes the exclusion constraint: there are no gluts allowed. The 'meaningless' approach considered in this chapter allows for glutty sentences as well as gappy sentences. As

for what to do with non-glutty sentences, the two approaches agree. We can see the 'meaningless' approach as standing in the same relation to the Weak Kleene approach as FDE stands in to K3.

6. Fill out the 'meaningless' approach (see §12.4) by adding predicates, quantifiers, and a necessity operator. (NB: there may be more than one way of doing this that is consistent with the basic 'meaningless' idea.)

   Answer. As the question mentions, there will be many different ways of doing this. One natural way is to simply use the truth- and falsity-conditions for predicates, quantifiers, and modalities that were used earlier in the book. They make just as much sense when combined with the 'meaningless' approach to $\wedge$ and $\vee$, so that could do.

   On the other hand, there is a natural sense in which $\wedge$, $\forall$, and $\Box$ are linked together, in Mfde, and this approach would break some of that connection. Think about Mfde for a moment. In that system, $\forall x F(x)$ always has the same value as $F(a) \wedge F(b) \wedge \ldots \wedge F(z)$, if $a, b \ldots z$ are all the names. That is, one is true at a world in a case iff the other is, and one is false at a world in a case iff the other is. (If there are infinitely many names, the version with $\wedge$ isn't a legitimate sentence—it's too long—but if it were the point would still hold.) This is a sensible feature: saying that everything is F is very like saying this thing is F and that thing is F and ... the last thing if F, too. If we take the 'meaningless' approach to $\wedge$, and leave $\forall$ alone, this connection will be broken. We can restore it by using the following truth- and falsity-conditions:

   - $[c, w] \models_1 \forall x A$ iff $[c, w] \models_1 A(\alpha/x)$ for all α such that $\delta(\alpha) \in E_w$.
   - $[c, w] \models_0 \forall x A$ iff 1) $[c, w] \models_! A(\alpha/x)$ for all α such that $\delta(\alpha) \in E_w$, and 2) $[c, w] \models_0 A(\alpha/x)$ for some α such that $\delta(\alpha) \in E_w$.

   Similarly, there is a connection between $\wedge$ and $\Box$ in Mfde that is lost if we modify $\wedge$ but not $\Box$. $\Box A$ is like a 'cross-world conjunction' in Mfde, and we can preserve that connection on the 'meaningless' approach with the following truth- and falsity-conditions:

   - $[c, w] \models_1 \Box A$ iff every $w' \in \mathcal{W}$ is such that $[c, w'] \models_1 A$.
   - $[c, w] \models_0 \Box A$ iff 1) every $w' \in \mathcal{W}$ is such that $[c, w'] \models_! A$, and 2) some $w' \in \mathcal{W}$ is such that $[c, w'] \models_0 A$.

   To maintain the connections between $\vee$, $\exists$, and $\Diamond$, similar changes would be necessary as well. The results:

   - $[c, w] \models_1 \exists x A$ iff 1) $[c, w] \models_! A(\alpha/x)$ for all α such that $\delta(\alpha) \in E_w$, and 2) $[c, w] \models_1 A(\alpha/x)$ for some α such that $\delta(\alpha) \in E_w$.
   - $[c, w] \models_0 \exists x A$ iff $[c, w] \models_0 A(\alpha/x)$ for all α such that $\delta(\alpha) \in E_w$.
   - $[c, w] \models_1 \Diamond A$ iff 1) every $w' \in \mathcal{W}$ is such that $[c, w'] \models_! A$, and 2) some $w' \in \mathcal{W}$ is such that $[c, w'] \models_1 A$.
   - $[c, w] \models_0 \Diamond A$ iff every $w' \in \mathcal{W}$ is such that $[c, w'] \models_0 A$.

One can use these modified clauses in any combination. There are certainly other options as well.

7. Invoking the definitions of 'contingent' and 'broadly contingent' from Chapter 5 exercises, give what you think are the right truth- and falsity-in-a-case conditions for a *contingency* operator in the otherwise Mfde setting. (In other words: add new unary connectives to serve as an *it is contingent that...* and *it is broadly contingent that...* operators in the otherwise Mfde setting. What is the logic of your connective(s) like? Explore!

   Answer. Here are some natural choices of truth- and falsity-conditions for connectives C (*it is contingent that*) and C_B (*it is broadly contingent that*):

   - $[c, w] \models_1 CA$ iff 1) there is a world $w' \in \mathcal{W}$ such that $[c, w'] \models_1 A$ and there is a world $w'' \in \mathcal{W}$ such that $[c, w''] \models_0 A$.
   - $[c, w] \models_0 CA$ iff either 1) every world $w' \in \mathcal{W}$ is such that $[c, w'] \models_1 A$ or 2) every world $w' \in \mathcal{W}$ is such that $[c, w'] \models_0 A$.
   - $[c, w] \models_1 C_B A$ iff 1) there is a world $w' \in \mathcal{W}$ such that $[c, w'] \models_1 A$ and there is a world $w'' \in \mathcal{W}$ such that $[c, w''] \not\models_1 A$.
   - $[c, w] \models_0 C_B A$ iff either 1) every world $w' \in \mathcal{W}$ is such that $[c, w'] \models_1 A$ or 2) every world $w' \in \mathcal{W}$ is such that $[c, w'] \not\models_1 A$.

   Other choices might work as well. Given the above definitions, added to Mfde, we have the following validities (among others):

   - $CA \dashv\vdash \Diamond A \wedge \Diamond \neg A$
   - $\neg CA \dashv\vdash \Box A \vee \Box \neg A$
   - $C_B A \vdash \Diamond A$
   - $\Box A \vdash \neg C_B A$
   - $\vdash C_B A \vee \neg C_B A$
   - $C_B A \wedge \neg C_B A \vdash B$

8. How might you add a *necessarily consistent* connective to the Mfde? What should the truth conditions for *it is necessarily consistent...* be in a broad Mfde setting? What about defining our target 'necessarily consistent' operator in Mfde thus: just let $\mathbb{C}A$ abbreviate $\Box\neg(A \wedge \neg A)$. What, then, is the logic of $\mathbb{C}$ in Mfde? Are there Mfde cases in which $\mathbb{C}A$ is a *glut*—and, so, true but itself 'inconsistent' (since also false)? Is this a problem for a consistency operator?

   Answer. Given the definition of $\mathbb{C}$ given in the question, we have the following validities (inter alia) in Mfde:

   - $\Box(A \vee \neg A) \dashv\vdash \mathbb{C}A$
   - $\Diamond(A \wedge \neg A) \dashv\vdash \neg\mathbb{C}A$
   - $\mathbb{C}A \dashv\vdash \mathbb{C}\neg A$

   There will definitely be cases c such that $[c, @] \models_1 \mathbb{C}A \wedge \neg\mathbb{C}A$: consider a case c with just one world, @, and let $[c, @] \models_1 p$ and $[c, @] \models_0 p$. In this case, we have $[c, @] \models_1 \Box(p \vee \neg p)$, and so $[c, @] \models_1 \mathbb{C}p$. We also have $[c, @] \models_1 \Diamond(p \wedge \neg p)$, and so $[c, @] \models_1 \neg\mathbb{C}A$. Thus, $[c, @] \models_1 \mathbb{C}p \wedge \neg\mathbb{C}p$.

On its own, this might not seem like too much trouble for a necessary consistency operator like $\mathbb{C}$. After all, the reason we introduce a necessary consistency operator is because we recognize that some things might not behave consistently; once we make this recognition, it doesn't seem like there's any particular reason that necessary consistency claims themselves must behave consistently.

On the other hand, there is something that might well seem off about this necessary consistency operator anyway. It lies not in the possibility of a case making $\mathbb{C}A \wedge \neg\mathbb{C}A$ true, but rather in the first two entailments presented above. As we can see, the first entailment tells us that any sentence that is never *gappy* is necessarily consistent, according to this operator. But then $\mathbb{C}$ seems more like a necessary *completeness* operator, and less like a necessary consistency operator. On the other hand, the second entailment tells us that any sentence that is anywhere *glutty* is not necessarily consistent; this is closer to what we might expect.

We might avoid the suggestion of completeness by offering something like the following truth- and falsity-conditions:

- $[c, w] \models_1 \mathbb{C}A$ iff no $w' \in \mathcal{W}$ is such that $[c, w'] \models_1 A \wedge \neg A$.
- $[c, w] \models_0 \mathbb{C}A$ iff some $w' \in \mathcal{W}$ is such that $[c, w'] \models_1 A \wedge \neg A$.

These falsity conditions are equivalent to those of the $\mathbb{C}$ defined in the question, but the truth conditions differ. One advantage of this new definition is that it seems to allow a restricted form of modus ponens: we have $\mathbb{C}A, A, A \to B \vdash B$. After all, as we've seen, failures of modus ponens rely on gluts. On the other hand, on this new definition, we can never have a case c and a world w such that $[c, w] \models_1 \mathbb{C}A \wedge \neg\mathbb{C}A$. If we're considering logics with true contradictions, it's not clear why we wouldn't want to allow them here.

(What if we allow contradictions into the language we use to *describe* our cases? That is, what if we consider cases c and worlds w such that $[c, w] \models_1 A$ and $[c, w] \not\models_1 A$? This kind of approach is well beyond the scope of this book, but seems to recommend itself here.)

9. What other phenomena might motivate different logics? Think and explore!

   Answer. This one's wide open.

Sample answers

Answer 2. Yes, $\Box A$ implies ∇A in the Mfde theory (expanded with the given actuality operator). To see this, consider any case c such that $[c, @] \models_1 \Box A$, in which case—by the Mfde truth conditions for the box—every world w in the given universe (case) is one at which A is true, that is, $[c, w] \not\models_1 A$ (for all w

in the universe). Hence, in particular, A is true at @, that is, $[c, @] \models_1 A$. But, then, given the truth conditions for ∇, we have that $[c, @] \models_1 \nabla A$. What we've shown here is that any (Mfde) case in which $\Box A$ is true is one in which ∇A is true (given the going truth conditions for ∇). What we've shown, in other words, is that $\Box A$ implies ∇A.

13

Tableau systems

13.1 What are tableaux?

To ask whether an argument from $A_1, A_2, \ldots, A_n$ to B is *valid* is to ask whether there is a case in which the As are all true and B is not. Tableaux provide a simple way to reason through questions like this. They can be seen as operating by *reductio*: we start by assuming that there is such a case, and reason through what else would have to hold if this were so. If we find that, no matter what, any case meeting the initial supposition would have to meet conditions that no case can meet, we conclude that there can be no such case, and so the argument is valid. On the other hand, if we find that there can be such a case, we conclude that the argument is valid.

It takes three steps to specify a tableau system. First, we need to say how to *start* a tableau: how to record the assumption that there is a counterexample to the argument in question. Next, we need to say how to *extend* a tableau: how to tell, given that there is a case meeting certain conditions, what additional conditions follow. And finally, we need to say when a tableau *closes*: how to tell when no case can meet the conditions imposed by a tableau.

In fact, tableau have a tree-like structure, with the start (the root) at the top. Sometimes, while extending a tableau downwards, we know if there's a case meeting conditions α, then there must be one meeting *either* conditions β or conditions γ, but we don't know which. When this happens, the tableau *branches*; on one branch, we add requirements β and see what follows, and on the other, we add requirements γ and see what follows. This can be seen in Figure 13.1. If we find that no case can meet the requirements on a branch, we close that branch indvidually, and only pursue the other branch. In Figure 13.1, the leftmost branch has closed (marked with $\times$), while the rightmost is still open. If no case can meet either set of requirements, then no case can meet requirements α.

This way of explaining tableaux makes them sound like a systematic way of reasoning about cases. Although they can be seen in this way, they don't have to be. Tableaux are fully specifiable without reference to cases, and they form a proof theory intimately related to more-obviously-proofy sequent calculi. (For details of the relation between tableaux and sequent calculi, see eg (Smullyan, 1995).)

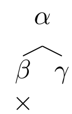

Fig. 13.1: A branching

$$\langle A_1 \quad \oplus\rangle$$
$$\langle A_2 \quad \oplus\rangle$$
$$\ldots$$
$$\langle A_n \quad \oplus\rangle$$
$$\langle B \quad \ominus\rangle$$

Fig. 13.2: The start of a tableau

13.2 Propositional tableaux

In this section, we explain how to build tableaux for the propositional systems in the book; these systems are presented in Chapters 4–6. The book proceeds by starting with classical cases, and considering a wider and wider variety of cases as it goes on. This section will proceed in reverse order: we'll give a tableau system that characterizes $\vdash_{FDE}$, and then add to it bit by bit as we go on, ending with a tableau system for $\vdash_{bc}$. The tableaux presented throughout this supplement closely follow those discussed in (Priest, 2008; Restall, 2005).

Our tableaux will be made up of *tagged nodes*: a tagged node is something of the form $\langle A \quad \circ\rangle$, where A is any sentence and $\circ$ is either $\oplus$ or $\ominus$. Here, $\oplus$ and $\ominus$ are the *tags*, and which tag appears in a tagged node will be important for deciding what to do with it. One way to think of the tags: a tagged node $\langle A \quad \oplus\rangle$ records that A is true in a case $\mathcal{M}$ ($\mathcal{M} \models_1 A$), and a node $\langle A \quad \ominus\rangle$ records that A isn't true in $\mathcal{M}$ ($\mathcal{M} \not\models_1 A$).

13.2.1 *Tableaux for $\vdash_{FDE}$*

13.2.1.1 *How to build a tableau*

Starting: Suppose we want to use tableaux to show that an argument from $A_1, A_2, \ldots, A_n$ to B is FDE-valid; that is, that $A_1, A_2, \ldots, A_n \vdash_{FDE} B$. This would mean that there is no case $\mathcal{M}$ in which all of $A_1, A_2, \ldots, A_n$ are true (that is, such that $\mathcal{M} \models_1 A_1$ and $\mathcal{M} \models_1 A_2$ and ...and $\mathcal{M} \models_1 A_n$) and B is not (that is, $\mathcal{M} \not\models_1 B$). So our tableau starts by assuming that there is such a case: we start with what's shown in Figure 13.2.

Extending: From there, we apply our rules, generating new nodes and branches from old. The rules for FDE tableaux are given in Figure 13.3. Each rule starts from a node on a branch, and it either adds nodes to the end of that branch, or it splits the end, turning the branch into two, and adds different nodes to each of the new branches. Each rule operates only on nodes of a certain form, but it's not important *where* on a branch the node is; you can apply a rule to the top node of a branch, any node you like in the middle,

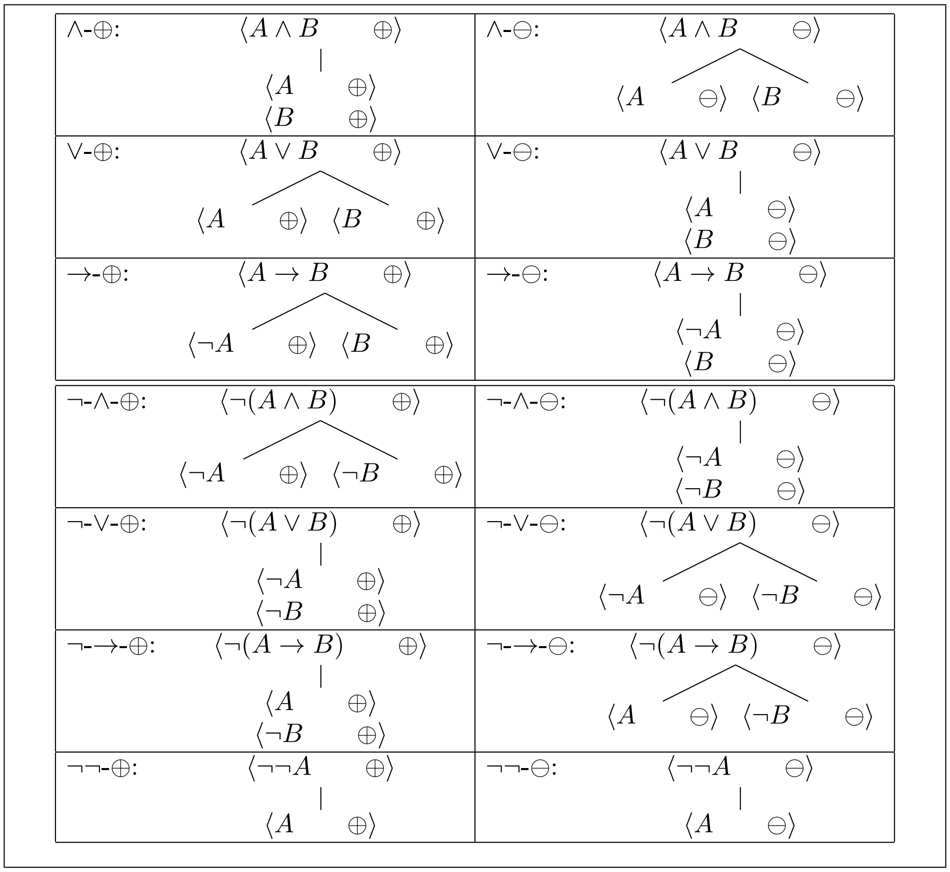

Fig. 13.3: Tableau rules

or the very bottom node. Whichever node you apply a rule to, add the new node(s) to the *bottom* of the branch.

Also, you can apply the rules in any order you like. However, it's usually a good idea to apply rules that don't branch before rules that do, so far as you can. This tends to keep tableaux shorter (or at least narrower).

Closing: A branch *closes* in an FDE tableau when it contains two tagged nodes of the form $\langle A \quad \oplus\rangle$ and $\langle A \quad \ominus\rangle$, for any sentence A. These would require that a case $\mathcal{M}$ was such that $\mathcal{M} \models_1 A$ and $\mathcal{M} \not\models_1 A$, but no case (as far as this book considers) can be like that. When a branch closes, we write $\times$ below it, and we don't bother applying any more rules to that branch. If every branch on a tableau closes, we say the tableau has closed, and the argument we used to generate the tableau is valid.

13.2.1.2 *An example* Figure 13.4 is an example FDE tableau, showing that $\neg(A \wedge B) \vdash_{FDE} \neg A \vee \neg B$. (This is one of the validities listed in §6.7 of the book;

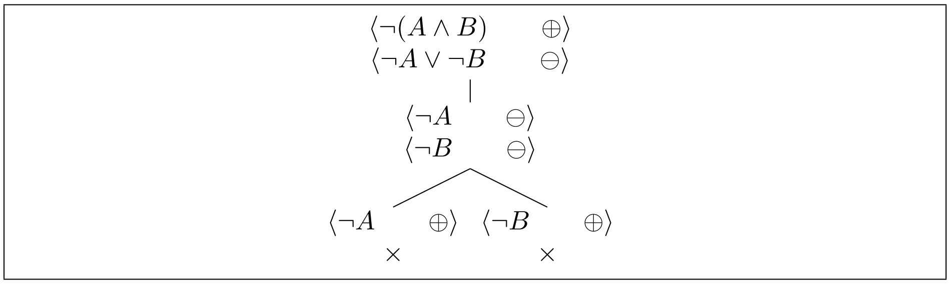

Fig. 13.4: Showing that $\neg(A \wedge B) \vdash_{FDE} \neg A \vee \neg B$

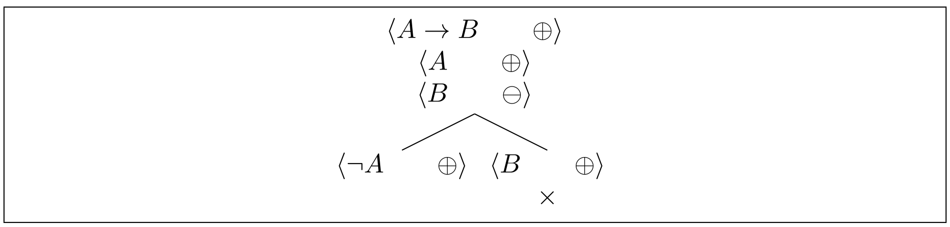

Fig. 13.5: Showing that $A \to B, A \nvdash_{FDE} B$

you are asked to consider it in Exercise 6.5.) In this figure, we've set things up in the appropriate way, and then applied two rules: first the $\vee$-$\ominus$ rule, and then the $\neg$-$\wedge$-$\oplus$ rule. At this point, both branches close, so the tableau is closed, and the argument is valid.

13.2.1.3 *When a tableau doesn't close* If a tableau built from an argument doesn't close, even after every rule that can be applied has been applied, then the original argument was invalid. Figure 13.5 provides an example, showing that $A \to B, A \nvdash_{FDE} B$. Once the beginning is in place, only one rule can be applied: $\to$-$\oplus$. This creates two branches. The right branch closes immediately, but the left one does not, and there is nothing left for us to do. We've applied every rule we can.

We can use an *open completed branch*, like the left branch in Figure 13.5, to find a case that counterexamples the original argument. Here, the branch tells us that we need to choose a case $\mathcal{M}$ such that $\mathcal{M} \models_1 \neg A$ (so $\mathcal{M} \models_0 A$), $\mathcal{M} \not\models_1 B$, and $\mathcal{M} \models_1 A$. Any such case will provide us with a counterexample.

13.2.2 *Tableaux for other logics*

A case like this, though, would not be a permissible paracomplete (non-paraconsistent) case. If we stick to the (paracomplete but not paraconsistent) cases of Chapter 5, no case $\mathcal{M}$ can be such that $\mathcal{M} \models_1 A$ and $\mathcal{M} \models_0 A$. Our tableau can rule this out with a simple modification: we add a different way for a branch to close. As before, any branch containing both $\langle A \quad \oplus \rangle$ and $\langle A \quad \ominus \rangle$ closes, but we also close any branch that contains both $\langle A \quad \oplus \rangle$ and $\langle \neg A \quad \oplus \rangle$. This rules out cases $\mathcal{M}$ such that $\mathcal{M} \models_1 A$ and $\mathcal{M} \models_0 A$, and so it gives us tableaux for the logic K_3. Return to Figure 13.5. The open left branch here would close if we applied this new rule; then the whole tableau would close. And indeed, $A \to B, A \vdash_{K3} B$.

Similarly, if we want to allow for inconsistent cases but not incomplete ones, we should not add the above closure condition, but rather a different one: that a branch closes if it contains both $\langle A \quad \ominus\rangle$ and $\langle \neg A \quad \ominus\rangle$. And if we want to rule out both inconsistent and incomplete cases, we can add both new closure conditions, resulting in tableau for $\vdash_{bc}$.

Parenthetical remark The tableau we arrive at by adding both new closure conditions have an interesting symmetry to them, which can be used to simplify them. It turns out that, with all three closure conditions in place, a branch with $\langle A \quad \ominus\rangle$ on it will close iff a branch just like it, but with $\langle \neg A \quad \oplus\rangle$ instead, closes. Similarly, a branch with $\langle A \quad \oplus\rangle$ on it will close iff a branch just like it, but with $\langle \neg A \quad \ominus\rangle$ instead, closes. This means we can stick entirely to either $\oplus$- or $\ominus$-tagged nodes, if we like, and use $\neg$ to simulate the behavior of whichever kind we don't use. If we do this, there is no need to write the tags at all anymore; we can simply use the bare sentences as nodes. The usual tableau presentation for classical logic does just this; it's equivalent to the $\oplus$-only version arrived at through this process. Without all three closure conditions in place, though, this equivalence doesn't work, which is why we make the tags explicit in tableaux for our nonclassical logics. *End parenthetical remark*

13.3 Predicate tableaux

Tableaux for predicate logic start, and are closed, in the very same way as tableaux for propositional logic. In particular, the difference between classical, paracomplete, and paraconsistent remains unchanged; that's simply a matter of which closure rules are in play. The difference between propositional and predicate tableaux is in the way predicate tableaux are extended.

13.3.1 *Unfree tableaux*

The rules for predicate tableaux, as you might expect, have a bit more to them than those for propositional tableaux. We don't need to change anything that's come so far; we just need to add some rules for dealing with $=$, $\forall$, and $\exists$. Figure 13.6 shows the shape of the new rules.

These rules require a bit of comment. First, note that there are no rules for nodes of the form $\langle a = b \quad \ominus\rangle$, $\langle \neg a = b \quad \oplus\rangle$, or $\langle \neg a = b \quad \ominus\rangle$. That's ok; the rules in place take care of $=$'s effects. Also, notice that the first rule for $=$ doesn't apply to any existing nodes; it generates nodes ex nihilo.

Some of these rules use terms t or u, while others use a term a. This is an important difference. In these rules, t and u can be any terms whatsoever that appear free on the branch. (If there are no terms free on the branch, then a new one can be introduced.) Where a appears (in the rules $\forall$-$\ominus$ and $\exists$-$\oplus$), it must be a term *new* to the branch; that is, a term that does not occur free anywhere else on the branch in question. Finally, note that a number of these rules can

$=$: $\quad .$ $\mid$ $\langle t = t \quad \oplus\rangle$	$=$-$\oplus$: $\langle t = u \quad \oplus\rangle$ $\langle A(t) \quad \oplus\rangle$ $\mid$ $\langle A(u) \quad \oplus\rangle$
$\forall$-$\oplus$: $\langle \forall x A(x) \quad \oplus\rangle$ $\mid$ $\langle A(t) \quad \oplus\rangle$	$\forall$-$\ominus$: $\langle \forall x A(x) \quad \ominus\rangle$ $\mid$ $\langle A(a) \quad \ominus\rangle$
$\exists$-$\oplus$: $\langle \exists x A(x) \quad \oplus\rangle$ $\mid$ $\langle A(a) \quad \oplus\rangle$	$\exists$-$\ominus$: $\langle \exists x A(x) \quad \ominus\rangle$ $\mid$ $\langle A(t) \quad \ominus\rangle$
$\neg$-$\forall$-$\oplus$: $\langle \neg\forall x A(x) \quad \oplus\rangle$ $\mid$ $\langle \exists x \neg A(x) \quad \oplus\rangle$	$\neg$-$\forall$-$\ominus$: $\langle \neg\forall x A(x) \quad \ominus\rangle$ $\mid$ $\langle \exists x \neg A(x) \quad \ominus\rangle$
$\neg$-$\exists$-$\oplus$: $\langle \neg\exists x A(x) \quad \oplus\rangle$ $\mid$ $\langle \forall x \neg A(x) \quad \oplus\rangle$	$\neg$-$\exists$-$\ominus$: $\langle \neg\exists x A(x) \quad \ominus\rangle$ $\mid$ $\langle \forall x \neg A(x) \quad \ominus\rangle$

Fig. 13.6: Predicate tableau rules

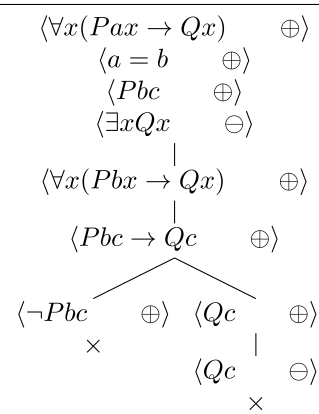

Fig. 13.7: Showing that $\forall x(Pax \to Qx), a = b, Pbc \vdash_{K3} \exists xQx$

be applied repeatedly, even to the very same nodes. For example, the $\forall$-$\oplus$ rule, since it has a t in it, can be applied once for every term that occurs free on the branch.

13.3.1.1 *Example* Figure 13.7 uses a tableau to show that $\forall x(Pax \to Qx), a = b, Pbc \vdash_{K3} \exists xQx$. Note that we've exercised some choices to keep the tableau small; as one example, we've applied $\forall$-$\oplus$ to get $Pbc \to Qc$. We could as well have used it to get, say $Pbb \to Qb$, but that wouldn't have helped close the tableau.

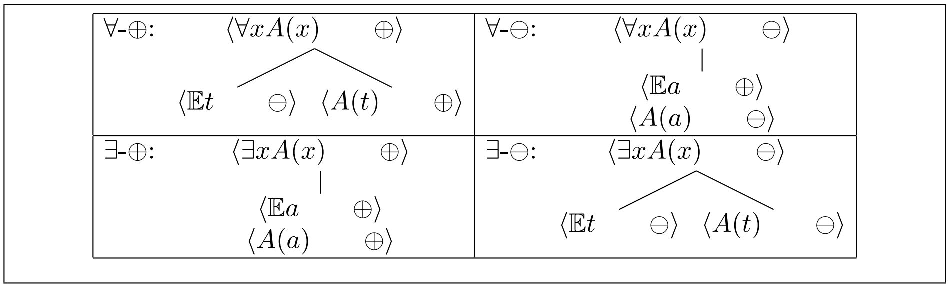

Fig. 13.8: Free logic tableau rules

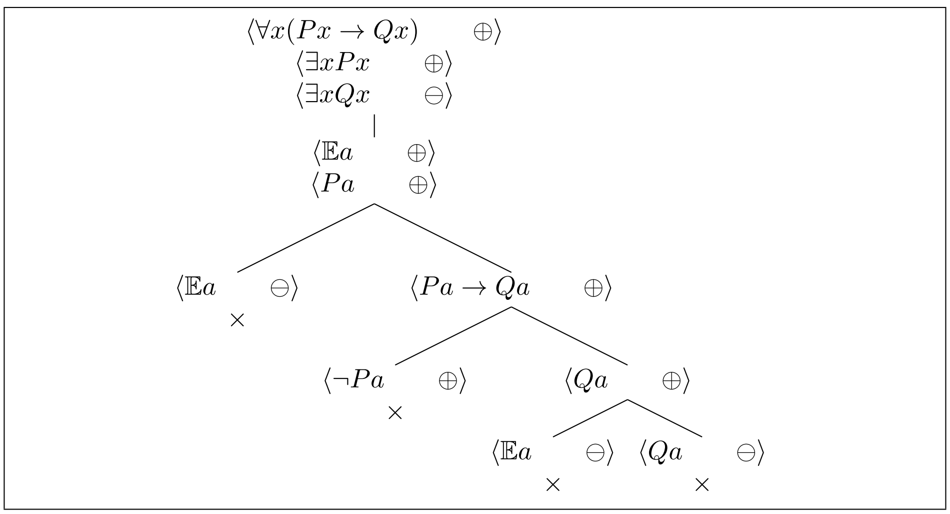

Fig. 13.9: Showing that $\forall x(Px \to Qx), \exists xPx \vdash \exists xQx$

13.3.2 *Free tableaux*

When it comes to tableaux for free logics, we have to handle the quantifiers slightly differently. Identity works just the same, as do the rules for negated quantifiers. We introduce a new predicate $\mathbb{E}$ to represent existence. It doesn't need to officially be in the language, but we'll use it in the course of these tableaux. Figure 13.8 gives the quantifier rules we need (here, a and t mean just the same as before):

13.3.2.1 *Example* Figure 13.9 shows that $\forall x(Px \to Qx), \exists xPx \vdash \exists xQx$ in 'freed up' paracomplete cases.

13.4 Modal tableaux

The nodes in modal tableau are slightly more complex. Instead of the form $\langle A \quad \circ\rangle$, they have the form $\langle A \quad w \quad \circ\rangle$, where w—the *world parameter*—is a number (standing in for a particular world in a case), and $\circ$ is again either $\oplus$ or $\ominus$. To begin a tableau, we start as before, making sure that all the starting nodes

$\Box$-$\oplus$:	$\langle \Box A \quad w \quad \oplus \rangle$	$\langle A \quad v \quad \oplus \rangle$	$\Box$-$\ominus$:	$\langle \Box A \quad w \quad \ominus \rangle$	$\langle A \quad i \quad \ominus \rangle$
$\Diamond$-$\oplus$:	$\langle \Diamond A \quad w \quad \oplus \rangle$	$\langle A \quad i \quad \oplus \rangle$	$\Diamond$-$\ominus$:	$\langle \Diamond A \quad w \quad \ominus \rangle$	$\langle A \quad v \quad \ominus \rangle$
$\neg$-$\Box$-$\oplus$:	$\langle \neg \Box A \quad w \quad \oplus \rangle$	$\langle \neg A \quad i \quad \oplus \rangle$	$\neg$-$\Box$-$\ominus$:	$\langle \neg \Box A \quad w \quad \ominus \rangle$	$\langle \neg A \quad v \quad \ominus \rangle$
$\neg$-$\Diamond$-$\oplus$:	$\langle \neg \Diamond A \quad w \quad \oplus \rangle$	$\langle \neg A \quad v \quad \oplus \rangle$	$\neg$-$\Diamond$-$\ominus$:	$\langle \neg \Diamond A \quad w \quad \ominus \rangle$	$\langle \neg A \quad i \quad \ominus \rangle$

Fig. 13.10: Rules for $\Box$ and $\Diamond$

share the same world parameter. (Whichever parameter this is, it's standing in for @ in the counterexample we're supposing exists.)

All the previous rules remain as unchanged as possible; they carry along their world parameter, but don't use it in any way. In our only two-premise rule—=-$\oplus$—we don't require the world parameters to match; there, the new node inherits the parameter from the $A(t)$ premise node. We add new rules to deal with $\Box$ and $\Diamond$; they are given in Figure 13.10. In these rules, v and w can stand in for any parameter at all, and i must be a parameter that doesn't occur anywhere on the branch yet. (Thus, v and w behave like t and u in our predicate rules, and i behaves like a.)

A branch closes when the closure conditions (whichever are in force) are met by two nodes that share the *same* world parameter.

13.4.0.2 *Example* In Figure 13.11, a tableau is used to show that $\Box\forall x(Px \wedge Qx) \vdash_{Mfde} \Diamond\exists x(Px \vee Rx)$. Note that where we introduce the term a, no term has yet occurred free on the branch, so we introduce a new term via the $\exists$-$\ominus$ rule.

$$
\begin{array}{c}
\langle \Box\forall x(Px \wedge Qx) \quad 1 \quad \oplus\rangle \\
\langle \Diamond\exists x(Px \vee Rx) \quad 1 \quad \ominus\rangle \\
| \\
\langle \exists x(Px \vee Rx) \quad 1 \quad \ominus\rangle \\
| \\
\langle Pa \vee Ra \quad 1 \quad \ominus\rangle \\
| \\
\langle Pa \quad 1 \quad \ominus\rangle \\
\langle Ra \quad 1 \quad \ominus\rangle \\
| \\
\langle \forall x(Px \wedge Qx) \quad 1 \quad \oplus\rangle \\
| \\
\langle Pa \wedge Qa \quad 1 \quad \oplus\rangle \\
| \\
\langle Pa \quad 1 \quad \oplus\rangle \\
\langle Qa \quad 1 \quad \oplus\rangle \\
\times
\end{array}
$$

Fig. 13.11: Showing that $\Box\forall x(Px \wedge Qx) \vdash_{Mfde} \Diamond\exists x(Px \vee Rx)$

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