Logic

Historically there has been considerable variation in the understanding of the scope of the field of logic. Our concern is with what may be called, to distinguish it from other conceptions, *formal, deductive* logic. Today, once one gets beyond the introductory level, it is customary to divide the field between mathematical and philosophical logic, each with subdivisions. But let us begin with the basics, and note two points. First, (deductive) logic has always had at its core a question equally relevant to philosophical dialectic and mathematical demonstration: What follows from what? Second, (formal) logic answers by pointing to and only to argument *forms*. Here whether a given conclusion is deducible from or a consequence of given premises, or whether the argument from premises to conclusions is valid, is taken to depend only on their forms. Logic proper is to be distinguished from history of logic and philosophy of logic, but one question from philosophy of logic must be mentioned at the outset: Should premises and conclusions be understood to be sentences or propositions expressed thereby? Is form a matter of composition of sentences using certain items of vocabulary, or composition of propositions? To this day the most elementary part of classical logic goes by the rival names of 'sentential logic' and 'propositional logic'. For present purposes, where it makes a difference we will use the former terminology, though what we have to say could mostly be reworded in the latter; some of the works cited use the one terminology, others the other. Also, one fact from the history of logic must be mentioned: Since some time in the first half of the twentieth century, when the traditional logic of syllogisms going back to Aristotle was finally subsumed and superseded, introductory textbooks have generally taught a common view, now known as *classical (elementary or first-order)* logic, as to which forms are valid.

Classical logic

Classical sentential logic considers form resulting from composition of sentences out of sentences, using such connectives as 'not', 'and', 'or' (negation, conjunction, disjunction); classical predicate logic considers also composition out of predicates and other subsentential components, using the quantifiers 'all' and 'some' (universal and existential). A conclusion is counted as a consequence of a set of premises if their form of composition alone guarantees that if the premises are true, so is the conclusion. The notion is made more rigorous by using symbolic formulas to represent forms; for 'someone loves everyone' and 'everyone loves someone' these might be $\exists x \forall y Fxy$ and $\forall x \exists y Fxy$. A model is defined to be a universe for variables x and y to range over (perhaps the set of all persons) plus a specification for each predicate letter F of what relation on the objects in the universe it stands for (perhaps that of lover to beloved). A rigorous definition of what it is for a formula to be true in a model is provided, and consequence defined in terms thereof: $\forall y \exists x Rxy$ is a consequence of $\exists x \forall y Rxy$ because the former is true in every model in which the latter is true. A proof or deduction is, roughly speaking, a break-down of the route from premises to conclusion into a sequence of short steps each of one of a few kinds. Different textbooks provide different proof-procedures, but all share two features: if there is a deduction then the conclusion is a consequence of the premises (soundness), and conversely if the conclusion is a consequence of the premises there is is a deduction (completeness). The 'semantic' notion of consequence coincides with the 'syntactic' notion of deducibility. (Texts differ over whether they include a demonstration of these 'metatheorems', and over what supplementary material if any they include on informal logic or critical thinking, inductive logic or elementary probability, and so on.) Kneale and Kneale 1962, a work of history of logic rather than logic proper, gives an account of the long history of syllogistic logic and the slow emergence of the classical logic that replaced it. Cohen and Nagel 1934 represents the transition. Hilbert and Ackermann 1928, Tarski 1941, Quine 1950, Church 1956, Suppes 1957 are textbooks of classical logic pioneering in their day, representing different kinds of proof procedures. The subject can be learned today from any number of excellent contemporary textbooks, among which we will not play favorites.

Church, Alonzo, *Introduction to Mathematical Logic*, Princeton: Princeton University Press, 1956.

A classic textbook, differing from Hilbert and Ackermann by still using notations derived from Russell, though both books use an axiomatic presentation in which certain logical laws are taken as axioms and all others derived from them by repeated application of a few simple rules.

Cohen, Morris Rapahel and Ernest Nagel, *An Introduction to Logic and Scientific Method*. New York: Harcourt Brace, 1934.

A long-influential textbook, half on formal logic, half on inductive logic, interesting as representing the period when classical logic was still in the process of displacing syllogistic logic in elementary college instruction; it has gone through several editions.

Hilbert, David and Wilhelm Ackermann, *Grundzüge der theoretischen Logik*, Berlin: Springer, 1928. Translated by G. G. Lekie and F. Steinhardt, with notes by R. E. Luce, as *Principles of Mathematical Logic*. , New York: Chelsea, 1950.

The first textbook in classical first-order logic, noted for raising the questions of completeness and decidability that were solved in subsequent work of Gödel and Church and Turing; it has gone through several editions in German and English.

Jeffrey, Richard C., *Formal Logic: Its Scope and Limits*. New York: McGraw Hill, 1967.

The first textbook to use the tree method, derived from E. W. Beth's 'semantic tableaux', as its proof procedure, notable for how far it goes into metatheory for an introductory text; it has gone through several editions.

Kneale, William and Martha Kneale, The Development of Logic, Oxford: Clarendon Press, 1962.

A work on the history of logic that, though dated in parts owing to an explosion of scholarship in recent decades, remains the best available panoramic overview; it is mentioned despite not belonging to logic proper because, after an account of the long history of syllogistic logic, it illuminatingly treats the gradual emergence of what has become classical logic from the time of George Boole onwards.

Quine, W. V. O., *Methods of Logic*. New York: Holt, 1950.

The first attempt at a textbook presentation of a version of Jaśkowski's natural deduction proof procedure (see under Proof theory below) for classical logic; it has gone through several editions.

Suppes, Patrick, *Introduction to Logic*. Princeton: Van Nostrand/Reinhold Press, 1957.

An early and influential presentation of classical logic with a natural-deduction proof procedure in the style of Gentzen (see under Proof theory below); it has gone through several editions and been much imitated.

Tarski, Alfred, *Introduction to Logic: and to the Methodology of Deductive Sciences*. Oxford: Oxford University Press, 1941.

Another early textbook using an axiomatic proof procedure, covering not only pure logic but the axiomatization of the theories of various kinds of numbers in mathematics; notable as a product of the figure many consider the second-greatest logician of the twentieth century (after Gödel), it has gone through several editions.

Philosophical logic

Philosophical logic as understood today (not to be confused with philosophy of logic or with the branch of philosophy of language formerly called 'philosophical logic') is conventionally divided into extraclassical and anticlassical, concerned respectively with extensions of and alternatives to classical logic. But be warned: it is not always easy to say whether a given logic represents a different theory on the same range of questions as classical logic, or a separate theory on a different range of questions. Of the logics given separate consideration here, tense and modal logic are generally taken to be extra-classical and intuitionistic, relevance/relevant, and paraconsistent logic to be anti-classical; conditional logic is harder to classify. Extraclassical logics differ greatly not only in what classical principles they reject, but also in their motivations for rejecting them. Free logic rejects rejects the claim that 'Something Fs' follows from 'Everything Fs' on the grounds that logical purity requires abstention from any existence assumptions, even the nontriviality assumption that the universe is nonempty. Quantum logic rejects the claim that 'either both p and q or both p and r' follows from 'both p and either q or r' on the grounds that it has been discovered empirically that a particle may both have a certain position and one or the other of two momenta, though it cannot have both a definite position and a definite momentum. Non-monotonic logic rejects the inference from 'C follows to A' to 'C follows from A and B', on the grounds that 'X flies' may be presumed given 'X is a bird' but not given also 'X is a penguin'. Free and quantum logics are generally regarded as anti-classical, but non-monotonic logic as extra-classical, concerned not with formal deductive validity but with a different relationship, warranted presumption in the absence of further evidence. But such judgments are debatable. Gabby and Guenthner 1983-1989 is a massive compendium covering far more varieties of philosophical logic than we have space even to mention here. Zalta has similarly wide coverage. Beall and Van Fraassen 2003, Priest 2008, Burgess 2009, are textbooks differing in scale and range of logics covered, each giving an idea of the diversity of the field. Shapiro 2007 and Haack 1996 belong more to philosophy of logic than philosophical logic, but give attention to several varieties of the latter.

Burgess, John P., *Philosophical Logic*, Princeton: Princeton University Press, 2009.

An introductory textbook, concisely treating (owing to the strict word limits of the series in which it appears) tense, modal, conditional, relevance/relevant, and intuitionistic logics; additional exercises are available from the author's personal homepage.

Beall, J. C. and Bas C. van Fraassen, *Possibilities and Paradox: An Introduction to Modal and Many-Valued Logic.*

A friendly introduction to standard philosophical logics at the propositional level, intended for those with no prior exposure to nonclassical logics. The book uses simple tagged tableau procedures, while focusing heavily on the semantic or model-theoretic motivations for such logics.

Gabbay, Dov and Franz Guenthner, eds. Handbook of Philosophical Logic, 4 vols. Dordrecht: D. Reidel Publishing Co., 1983-1989.

An important collection of surveys, with the first, second, and third volumes covering respectively classical logic, extenstions thereof, and alternatives thereto, these last including discussion of logics we have had no space to address separately; the work is in the process of expansion to a second edition already up to over a dozen volumes and counting.

Haack, Susan, *Deviant Logic, Fuzzy Logic: Beyond the Formalism*. Chicago: University of Chicago Press, 1996.

A work, belonging to philosophy of logic, that nonetheless — because its thesis is that classical logic is revisable, but none of the anticlassical logics on the market would be a good revision — includes substantial material in philosophical logic in the form of a survey of many nonclassical logics, with special attention to the kind of many-valued logic called *fuzzy*.

Priest, Graham, *An Introduction to Non-Classical Logic: From Ifs to Is*, 2nd ed.. Cambridge: Cambridge University Press, 2008.

In its second edition, a weighty tome treating at length modal, tense, conditional, intuitionistic, many-valued, paraconsistent, 'relevant', and fuzzy logics.

Shapiro, Stewart editor, *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford: Oxford University Press, 2007.

A work on the philosophy of logic, in the form of a collection of surveys, mostly by prominent figures, of the pros and cons of various positions in philosophy of mathematics and of logic; it is mentioned despite not belonging to logic proper because it overlaps with philosophical logic by touching on a variety of anticlassical logics, especially intuitionistic and relevance/relevant.

Zalta, Edward, editor, *Stanford Encyclopedia of Philosophy* [http://plato.stanford.edu/contents.html]*

Provides introductory surveys, with ample references to further literature, for all logics specifically mentioned here, and others we have had no space for, including deontic, epistemic, fuzzy, linear, many-valued, provability, and dynamic logics.

Modal logic

The paradigmatic case of an extraclassical logic is the logic of the *modalities*, necessity and possibility, the prototype of a logic that adds to classical logic extra-classical connectives of philosophical interest, in this case 'necessarily' and 'possibly' (sometimes given variant readings such as the *deontic* 'obligatorily' and 'permissibly' or the *epistemic*). It was first treated at book length in the modern era in Lewis and Langford 1932. The type of models first developed for work with this logic have found applications to many other kinds as well. Kripke 1963 is an accessible primary source for this type of model, which Goldblatt 2006 sets in its historical context. The subject is covered in all the general references for philosophical logic listed above, and at much greater length and in much greater detail in Blackburn et al. 2002, which illustrates how the subject has become of interest outside philosophy, especially in theoretical computer science. Cresswell 1990 and Boolos 1993 deal with specialized issues of philosophical interest.

Blackburn, Patrick and Maarten de Rijke and Yde Venema, *Modal Logic.* Cambridge: Cambridge University Press, 2002.

A comprehensive textbook, directed towards mathematically sophisticated readers.

Boolos, George, *The Logic of Provability*. Cambridge: Cambridge University Press, 1993.

The definitive work on the specialized branch of modal logic introduced by Gödel, in which the necessity involved is interpreted as formal provability.

Cresswell, Max, *Entities and Indices*, Dordrecht: Kluwer, 1990.

A study of the comparative expressive power of modal logic as conventionally formulated and classical first-order logic with explicit quantification over possible worlds.

Goldblatt, 'Mathematical modal logic: a view of its evolution.' *Journal of Applied Logic* 1 (2006): 309-92.

A magesterial account of the history of the technical side of modal logic, among other things setting the work of Kripke in its historical context.

Kripke, Saul, 'Semantical Considerations on Modal Logic.' *Acta Philosophica Fennica* 16 (1963): 83-94.

The *locus classicus* for 'possible worlds semantics', part of a large body of related work by its author that opened the door to a great deal of philosophical activity in and around modal logic, appearing in a journal number with other papers from a memorable conference on modal logic in Helsinki in 1962.

Lewis, Clarence I. and Cooper H. Langford, *Symbolic Logic*, New York: Century Publishing, 1932.

The first book-length exposition of the subject by the founder of modern modal logic (viz. Lewis), in which modal logic is presented as if *in opposition* to classical logic, especially in the discussion of the so-called paradoxes of 'material implication'; available in an undated reproduction from Dover.

Tense logic

Temporal or tense logic adds to classical logic such connectives of time-reference such as 'it (once) was the case that' and 'it (sometime) will be the case that'. Philosophers who have strong views about what notions are properly viewed as logical may find these logics to be 'not really logic', regarding temporal logic as a kind of physics, as they may even regard modal logic as a kind of metaphysics (at least when the modalities are understood as 'metaphysical' necessity and possibility). By contrast, setting philosophy aside mathematicians and computer scientists generally call anything 'logic' that can be fruitfully studied by adaptations of techniques traditionally used in connection with assessment of the (logical) validity of arguments, just as they will call anything 'geometry' that can be studied by adaptations of techniques traditionally used in connection with the analysis of the structure of space, in either case regardless of originally intended applications if any. Prior 1967 is the pioneering work that put the subject on the map. Van Benthem 1991 is substantial textbook, providing more detail than in the general references on philosophical logic listed earlier. Burgess 1980 and Goldblatt 1980 are studies of special topics in the field of philosophical and mathematical interest. Pnueli 1977 inaugurated the application of the subject within computer science.

Benthem, Johann van, *Logic and Time*, Dordrecht: Kluwer, 1991.

A textbook covering not only tense logic, but the treatment of time-reference within classical logic as well.

Burgess, John P., 'Decidability for branching time.' *Studia Logica* 39 (1980): 203-18.

A more formal treatment of the philosophical issue of future contingents than is discussed in Prior's classic work (see below); shows how modal and tense operators interact when combined.

Goldblatt, Robert, 'Diodorean modality in Minkowski spacetime.' *Studia Logica* 39 (1980): 219-36.

A sophisticated case study of how tense logic reflects assumptions of theoretical physics.

Pnueli, Amir, 'The temporal logic of programs.' *Proceedings of the 18th IEEE Symposium on the Foundations of Computer Science*. New York: Institute of Electrical and Electronics Engineers, 1977: 46-57.

The paper, quite accessible to non-specialists, that loosed a flood of work by theoretical computer scientists in the area of tense logic.

Prior, Arthur N., *Past, Present and Future*. Oxford: Clarendon Press, 1967.

A classic text, by the founder of tense logic as a distinct branch of logic, on the philosophical motivations of tense logic and the early technical development of the subject; the first and perhaps still the most important book-length work in the area.

Conditional logic

Logics of subjunctive conditionals 'if p had been the case, then q would have been the case' are usually considered extraclassical, while logics of indicative conditionals 'if p is the case, then q is the case' are usually considered anticlassical if they do not equate the conditional with the disjunction 'either p is not the case or q is the case'. This classification reflects the standard thought that classical logic is committed to the given disjunctive analysis of indicatives but is silent on subjunctives. But the formalisms proposed in the two cases turn out to be similar in many ways, so that neither can be adequately studied without consideration of the other. Bennett 2003 covers both from a philosophical point of view. Adams 1975 and Edgington 2001 concern mainly indicatives, while Lewis 1973 concerns subjunctives, but the first-named work also shows the close connections between different formalisms. Hailperin 1996 is indirectly relevant.

Adams, Ernest, *The Logic of Conditionals: An Application of Probability to Deductive Logic.* Dordrecht: Reidel, 1975.

An account of a anticlassical theory of indicative conditionals, drawing on probability theory and the notion of conditional probability, but containing the remarkable result that the formalism obtained agrees with that proposed by Lewis for subjunctive conditionals, which uses a totally different kind of model.

Bennett, Jonathan, *A Philosophical Guide to Conditionals*, Oxford: Oxford University Press, 2003.

A philosophically-oriented account of work, intimately related with modal logic, on indicative and subjunctive conditionals alike, presenting results and examples of Ernest Adams, David Lewis, and others.

Edgington, Dorothy, 'Conditionals.' In *The Blackwell Guide to Philosophical Logic*. Edited by Lou Goble. Oxford: Blackwell, 2001, 385-414.

A concise account of the main positions in recent debates over the analysis of conditionals of various kinds.

Edgington, Dorothy, 'Indicative Conditionals', Stanford Encyclopedia of Philosophy. [http://plato.stanford.edu/entries/conditionals/]*

A concise account of various positions, focusing specifically on indicative as opposed to subjunctive conditionals.

Hailperin, Theodore, *Probability Logic: Origins, Development, Current Status, and Technical Applications*. Bethlehem, Pennsylvania: Lehigh University Press, 1996.

A work not directly on conditional logic, but inviting comparison with, and perhaps ultimately integration with, that of Adams.

Lewis, David, *Counterfactuals*. Oxford: Blackwell, 1973; revised ed. Cambridge, Massachusets: Harvard University Press, 1986.

An influential proposal in the logic of subjunctive conditionals, related to but distinct from proposals of Robert Stalnaker.

Intuitionistic logic

Moving on to more definitely anticlassical logics, the logic associated with mathematical intuitionism, a break-away movement dissenting from the direction in which pure mathematics was developing in the early twentieth century, holds that classical logic correctly represents the forms of argument accepted by classical mathematicians, but some of these forms of argument are incorrect. Among these are the argument from 'if p then q' and 'if not p then q' to 'q', which relies on the law of excluded middle 'p or not p', standardly understood to deliver that the disjunction of any (declarative) sentence and its negation is true. Underlying the rejection of this law, intuitionistic logic generally rejects the explanation of logical particles in terms of the conditions under which compounds formed with them are true. Instead of holding with classical logic that 'p or q' is true just in case 'p' is true or 'q' is true, intuitionistic logic holds a *proof* of 'p or q' consists of either a *proof* of 'p' or a *proof* of 'q'. As a result, the intuitionist will not endorse an instance of the classical law of excluded middle unless in possession either of a proof of 'p' or a refutation of 'p' (a proof of 'not p'). As to what underlies the rejection of truth-conditions in favor of proof-conditions, different intuitionists have offered different accounts, ranging from early approaches resting on idealist and mystical considerations tending towards solipsism, to later approaches based on verificationist considerations tending towards behaviorism. Brouwer 1976 represents the original approach, by the founder of intuitionism, and Heyting 1956 the somewhat more moderate views of his chief disciple, while Dummett 1973 and 2000 represent the later view, by the most prominent recent philosophical defender of intuitionism, and Prawitz 1977 an evaluation thereof by a sympathetic critic. The standpoint interested in metatheoretic questions without commitment to any intuitionistic philosophy is represented by Burgess 1981, building on work of Kreisel, while Martin-Löf represents the main present-day form of 'constructivist' mathematics making use of intuitionistic logic. Gödel 1986 links intuitionistic with modal logic.

Brouwer, L. E. J., *Collected Works*, vol. 1, *Philosophy and Foundations of Mathematics*. Edited by Arend Heyting. Amsterdam: North Holland, 1976.
A collection containing, spread over several key papers, Brouwer's presentation of his original ideas on the philosophical motivation of intuitionistic logic, to be compared with later accounts.

Burgess, John P., 'The completeness of intuitionistic propositional calculus for its intended interpretation.' *Notre Dame Journal of Formal Logic* 22 (1981): 17-18.
An adaptation, using Kripke models, of a method of Georg Kreisel to show that, on certain assumptions, every law of classical sentential logic that is intuitionistically acceptable is provable by the usual intuitionistic system.

Dummett, Michael, 'The philosophical basis of intuitionistic logic.' In *Logical Colloquium 73*. Edited by H. E. Rose and J. C. Sheperdson, Amsterdam: North Holland, 1973.
A classic paper, multiply anthologized, that advances arguments for applying intuitionistic logic well beyond the confines of mathematics.

Dummett, Michael, *Elements of Intuitionism*. Oxford: Oxford University Press, 2000.
A comprehensive introduction to intuitionistic logic and mathematics, with an appendix reiterating the author's distinctive philosophical case for intuitionism.

Gödel, Kurt, 'Eine interpretation des intuitionistischen Aussagenkalkuls.' Translated by John Dawson as 'An interpretation of intuitionistic propositional calculus.' Both on facing pages in *Collected Works of Kurt Gödel*, vol. 1. Edited by Solomon Feferman and others. Oxford: Oxford University Press, 1986, 300-303.
A miraculous paper in which, in three pages, Gödel indicates an interpretation of classical logic in intuitionistic logic and an interpretation of intuitionistic logic in modal logic, beside sketching a new and more convenient axiomatization of modal logic, and launching the new subject of provability logic.

Heyting, Arend, *Intuitionism*. Amsterdam: North Holland, 1956.
A short account of intuitionistic mathematics and its logic by the Brouwer disciple who first presented intuitionistic logic as an axiomatic system.

Martin-Löf, Per, *Intuitionistic Type Theory*. Naples: Bibliopolis, 1984.
The framework for the most active on-going program of constructivistic mathematics, making use of intuitionistic logic, a formalism for which computer scientists have produced several proof-assistant programs, and a modification of which lies at the base of currently fashionable 'univalent foundations'.

Prawitz, Dag, 'Meaning and proofs: the conflict between classical and intuitionistic logic,' *Theoria* 43 (1977): 2-43.
An assessment of attempts to apply ideas from proof theory to motivate a preference for intuitionistic over classical logic, by an eminent proof theorist.

Relevance/Relevant logic

What some call 'relevance' and others call 'relevant' logic rejects the classical doctrine that 'p and not p' logically implies an arbitrary 'q', on grounds of lack of a connection of relevance between premise and conclusion, and is soon led to reject also the inference form 'p or q and not p' to 'q'. Anderson, Belnap, et al. 1975/1992 is a major reference by pioneers of the subject, representing what has come to be called the 'American' approach, and Dunn and Restall 2002 a shorter survey. The 'Australian' approach is represented by Routley, Meyer, et al. 1983/2003, and the 'Scottish' by Read 1989, and a synthesis of 'American' and 'Australian' by Mares 2004, while Restall 2000 introduces a broader category of nonclassical logics subsuming relevance/relevant logic.

Anderson, Alan Ross and Nuel D. Belnap and J. Michael Dunn, *Entailment: The Logic of Relevance and Necessity,* vol. 2. Princeton: Princeton University Press. 1992.

The classic source for relevance/relevant logic, both the formal details (up to the time of publication) and the philosophical ideas involved, at least from the 'American' perspective. Comparison between the volumes shows the rapid growth in sophistication in the technical side of the subject (through contributions of Fine, Kripke, Meyer and Routley, Urquhart, and others).

Dunn, J.M. and Greg Restall, 'Relevance logic and entailment.' In *Handbook of Philosophical Logic*, vol. 6. Edited by F. Guenthner and Dov M. Gabbay, Dordrecht: Reidel, 2002, 1–128.

A panoramic survey of both philosophical and formal issues in relevance/relevant logic, written by one of the chief pioneers of the subject (viz., Dunn) and one of the leading experts in the broader field of substructural logics (see below).

Routley, Richard [later Sylvan, Richard] and Rober K. Meyer, Valerie Plumwood, and Ross Brady, *Relevant Logics and its Rivals* vol. 1. Atascardero, California: Ridgeview, 1983.

Brady, Ross, editor, *Relevant Logics and their Rivals*, voi. 2, Aldershot: Ashgate, 2003.

A resource for formal results on a variety of relevance/relevant logics. The philosophical remarks in the first volume are not representative of relevance/relevant logicians generally, but give a good example of certain philosophical perspectives in the field, especially in Australia. The second volume covers a wider range of topics including standard challenges for relevance/relevant logics such as conditionals and quantification, as well as a variety of metalogical results.

Mares, Edwin D., *Relevant Logic: A Philosophical Interpretation*. Cambridge: Cambridge University Press, 2004.

A recent volume advancing a blend of early 'American' and contemporary 'Australasian' philosophical perspectives in an effort to illuminate and defend the centrality of one particular system, Anderson and Belnap's R.

Read, Stephen, *Relevant Logic: A Philosophical Examination of Inference*. Oxford: Basil Blackwell, 1989.

A volume advancing what has come to be called the 'Scottish' perspective in relevant logic, containing a useful discussion of other perspectives (so-called American, Australian, Australasian), and highlighting many of the distinctions (and difficulties) involved in relevance/relevant logic.

Restall, Greg. An Introduction to Substructural Logic. London: Routledge, 2000.

A general treatment of logics that give up one or more of the standard ‘structural’ rules in a Gentzen-style formulation of classical logic, a class that includes relevance/relevant logics as well as other nonclassical logics, such as what is called *linear* logic, with varying philosophical and technical motivations.

Paraconsistent logic

On the paraconsistent view the two truth-values, true and false, need not be exclusive, and a single example (perhaps the liar paradox, ‘this very statement is false’) may have both. Often this is combined with the ‘paracomplete’ view that they need not be exhaustive, and that an example (perhaps the truth-teller pathology, ‘this very statement is true’) may have neither. But like classical logic and unlike intuitionistic logic, paraconsistent logic does allow that the truth-values of compounds are determined by those of their components, and that validity is a matter of form guaranteeing appropriate preservation of truth-values, from the more complicated set of possible truth-values recognized. Paraconsistent logic is perhaps the chief example of a *many-valued* logic that comes with a philosophical interpretation and motivation; many others are used mainly as technical tools. For nonclassical logics may turn out to have a utility that does not require acceptance of the original philosophical motivation for the logic as anything more than heuristically suggestive. Asenjo 1966 and Asenjo and Tamburino 1975 are pioneering works, and Priest 1979 the starting point for many subsequent efforts; Beall et al. 2014 compares their approaches. Routley 1979 represents an extreme view, while Routley and Meyer 1976 and especially Belnap 1977 point to the need for or utility of paraconsistent logic in a way that does not depend on assuming that there are literal contradictions that are literally true.

Asenjo, Florencio González, ‘Calculus for antinomies.’ *Notre Dame Journal of Formal Logic*, 7 (1966): 103-105.

A pioneering work of the glut-theoretic approach to paradoxes (i.e., treating paradoxical sentences as both true and false), introducing a propositional 3-valued paraconsistent logic, which is now more standardly called ‘Logic of Paradox’ or ‘LP’ based on wider familiarity with the work of Graham Priest (see below).

Asenjo, Florencio González and Joanna Tamburino, ‘Logic of antinomies.’ *Notre Dame Journal of Formal Logic*, 16 (1975): 17-44.

An extension of the glut-theoretic approach to paradox of Asenjo from sentential to predicate logic.

Beall, Jc and Michael Hughes and Ross Vandegrift, ‘Glutty theories and the logic of antinomies.’ In *The Metaphysics of Logic*. Edited by P. Rush, Cambridge: Cambridge University Press, 2014.

An elementary discussion of the formal and philosophical differences between the Asenjo-Tamburino logic of antinomies LA (see above) and the Priest first-order logic of paradox LP (see below).

Belnap, Nuel. ‘How a computer should think’.’ In *Contemporary Aspects of Philosophy*. Edited by Gilbert Ryle, Stockfield: Oriel Press, 1977, 30-56; reprinted as ‘A useful four-valued logic’ in Anderson, Belnap, and Dunn 1992 (see under Relevance/Relevant logic above), 506-41.

A classic example of a many-valued subclassical logic, and a classic example of applying paraconsistent logic without any suggestion of glut theory (i.e., entertaining the possibility of true negation-inconsistent theories).

Jaśkowski, Stanislaw. 'Rachunek zdań dla system dedukcyinych sprzecznych', *Studia Societatis Scientiarum Torunesis* Section A, 1:5 (1948): 55-77. English translation published as 'Propositional Calculus for Contradictory Deductive Systems', *Studia Logica* 24 (1969): 143-57.

This is the first clear construction of a paraconsistent logic, which was tied closely to philosophical issues surrounding vagueness. The idea takes different points of a model to be discussants; and truth in a model is truth at at least one such point. The points can all be consistent; but we can have models in which a sentence and its negation are true without all sentences being true in the model. (Jaśkowski's work has contributed greatly to contemporary ideas in philosophical logic, even though few philosophers have heard of him.)

Priest, Graham, 'The logic of paradox.' *Journal of Philosophical Logic* 8 (1979): 219-291,

A paper advocating a glutty approach to standard paradoxes generally, pioneering the argument from 'semantic closure' or 'expressive completeness' to gluts — sentences that are both true and false (i.e., have true negations) according to the given theory.

Routley, Richard, 'Dialectical logic, semantics, and metamathematics.', *Erkenntnis*, 14 (1979): 301-331.

A paper resembling Priest 1979 (see above) in advocating a paraconsistent logic for purposes of a glut-theoretic treatment of paradoxes, arguing that standard 'limitative theorems' are in fact arguments for gluts, a view which is not widely held even among glut theorists.

Routley, Richard and Robert K. Meyer,'Dialectical logic, classical logic, and the consistency of the world.' *Studies in Soviet Thought* 16 (1976): 1-25.

An early discussion of the alleged need for paraconsistent logic, pushing for a paraconsistent and paracomplete logic, while arguing for a philosophical view of 'agnosticism' about gluts and their dual 'gaps'.

Mathematical logic

Mathematical logic is conventionally divided into four distinct but interacting subfields, whose relations to basic logic and whose bearing on philosophy differ considerably from case to case: model theory, proof theory, set theory, and recursion theory (which many nowadays would rechristen 'computability theory'). Crossley et al. 1972 provides a concise overview, and Barwise 1977 a compendium of detailed topic-by-topic surveys. Hilbert and Bernays 1934/1937, Kleene 1952, and Shoefiled 1967 represent three generations of large-scale texts, indicative of the evolution of the subject over its first several decades. Van Heijenoort 1967 reprints in translation many fundamental papers, with expert commentary.

Barwise, K. Jon, ed. Handbook of Mathematical Logic. Amsterdam: North Holland, 1977.

A classic collection of high-level surveys, prototype for the many handbooks in more specialized areas that have appeared in recent decades.

Crossley, J. N. and C. J. Ash, N. H. Williams, and C. J. Brickhill, *What Is Mathematical Logic?* Oxford: Oxford University Press, 1972; reprinted New York: Dover, 1990.

A work of high-level popularization, explaining without pretending to prove important results from all branches of mathematical logic.

Hilbert, David and Paul Bernays, *Grundlagen der Mathematik* [*Foundations of Mathematics*], 2 vols. Berlin: Springer, 1934/37.

An absolute classic, a massive work presenting the results of multiple workers through the 1930s, not only on the ‘Hilbert program’, but on other areas of mathematical logic; a project for a English translation with the original German on facing pages is underway, with various partial translations being made available in the meantime.

Kleene, Stephen Cole, *Introduction to Metamathematics*, Amsterdam: North Holland, 1952.

Another large-scale work, by an important participant (along with Church and Turing) in the establishment of recursion theory, incorporating many of the results from the first two decades after Gödel’s fundamental work; the textbook from which mathematical logicians learned their subject for a generation and more.

Shoenfield, Joseph, *Mathematical Logic*. Reading, Massachusetts: Addison-Wesley, 1967.

A textbook at the level of beginning graduate students in mathematics, widely used ever since its appearance, and full of exercises.

Van Heijenoort, Jan, ed. From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879-1931, Cambridge: Harvard University Press, 1967.

A classic collection of translations of key documents by Frege, Gödel, and others, Brouwer, Cantor, Hilbert, Löwenheim, Russell, Skolem, Zermelo among them.

Model theory

The theory of models begins with the work of Alfred Tarski, reprinted in translation in Tarski 1983, from which emerged the definitions of model and truth-in-a-model used in the metatheory of classical logic. Tarski’s work also inspired (though a kind of reversal of perspective is involved) Davidsonian truth-conditional accounts of meaning, among other philosophical developments taking us outside the domain of logic. Kreisel 1969 is a philosophical reflection on the relation of technical notions of model theory and intuitive notions they attempt to capture. A number of what may be considered extraclassical logics, adding additional quantifiers rather than connectives to classical logic, work with the same Tarskian notion of model as classical first-order logic; surveys of this diverse range of logics may be found in Barwise and Feferman 1985. Väänänen 2007 is the definitive treatment of one such logic that has sparked considerable interest in philosophical circles. As for the model theory specifically of first-order logic, Robinson 1965 is an account by a pioneer in its application to abstract algebra of some early, elementary examples of such applications. Such applied model theory has since grown to enormous size, and represents the main thrust of model theory today, albeit not the part of the subject most inviting to philosophers. Ebbinghaus and Flum 1995 covers an important subfield not thoroughly treated in our other references, with applications in a different direction. Chang and Keisler 2012 is the latest version of a once widely-used textbook, covering the subject as it was before the revolution inaugurated by

Shelah, Saharon, Classification Theory and the Number of Non-Isomorphic Models, Amsterdam: North Holland, 1978.

Raised the subject to a level of mathematical sophistication and technical virtuosity where few philosophers can hope to follow.

Barwise, K. Jon and Solomon Feferman, eds. Model-Theoretic Logics. Berlin: Springer, 1985.

A collection of high-level surveys of logics that go beyond, but use the same notion of model as, classical logic, including logics of generalized quantifiers, infinitary logic, second-order logic, and more.

Chang, C. C. and H. Jerome Keisler, Model Theory, 3rd ed. Mineola, New York: Dover, 2012.

A much-used textbook pitched at an advanced undergraduate or beginning graduate level.

Ebbinghaus, Heinz-Dieter and Jörg Flum, Finite Model Theory. Berlin: Springer, 1995.

The first comprehensive textbook on the theory of finite models, which differs greatly in flavor from the theory of arbitrary (finite or infinite) models.

Kreisel, Georg. ‘Informal Rigor and Completeness Proofs’ abridged version. In The Philosophy of Mathematics. Edited by Jaakko Hintikka. Oxford: Oxford University Press, 1969, 78-94.

A subtle discussion of, among other things, the relationship between the technical notion of truth in all models and the intuitive notion of logical validity.

Robinson, Abraham. Introduction to Model Theory and to the Metamathematics of Algebra, 2nd ed. Amsterdam: North Holland, 1965.

An early and accessible textbook by a pioneer in the applications of model theory to abstract algebra, including a brief account of the author’s non-standard analysis, a modern version of infinitesimal calculus.

Tarski, Alfred. Logic, Semantics, Metamathematics: Papers from 1923 to 1938, 2nd ed. Translated by J. H. Woodger, Edited and with an Introduction by John Corcoran, Indianapolis: Hackett Publishing Co., 1983.

A collection including not only the celebrated ‘Concept of Truth in Formalized Languages’ but also other fundamental contributions to the creation of model theory.

Set theory

Georg Cantor created the theory of sets in the late nineteenth century, at first in connection with certain problems of mathematical analysis. Some of his key papers are made available in translation inf Cantor 1952, while what has become of Cantor’s original concerns with mathematical analysis can be see from Kechris 1995, and heterodox non-Cantorian approaches are surveyed in Randall 2014. The discovery of paradoxes led to a more rigorous, axiomatic treatment of the subject by Ernst Zermelo, for whose work see Zermelo 2010, and others. By the middle of the twentieth century it had been found that in some sense all of mathematics can be codified on the basis of the Zermelo-Fraenkel axiom system (ZFC). Set-theoretic terminology and results came to pervade mathematics, including numerous areas of interest in connection with the more formal parts of philosophy (including probability theory, to name just one). Kurt Gödel had by this time shown that in *any* axiom system for mathematics there will be questions that can be posed but not answered, and together with Paul Cohen he showed that for ZFC specifically, the conjecture of Cantor known as the *continuum hypothesis* (CH) was undecidable. The question whether and in what sense there can nonetheless be 'right' answers in such cases has occupied set-theorists and philosophers of mathematics ever since, beginning with Gödel himself in the work reprinted as Gödel 1990. Hrbacek and Jech 1999, and Jech 2003 are widely-used textbooks and the undergraduate and graduate levels, and Kanamori 2010 a survey of more advanced material.

Cantor, Georg. Contributions to the Founding of the Theory of Transfiinte Numbers, Translated and with an Introduction and Notes by Philip E. B. Jourdain, Mineola, New York: Dover Publishing Co., 1952.

A conveniently available photographic reproduction of a 1915 original, providing in English two major papers by the founder of set theory, still of more than historical interest, with useful supplements by the editor.

Gödel, Kurt. 'What Is Cantor's Continuum Problem?' in Collected Works, vol. 2. Edited by Solomon Feferman and others. Oxford: Oxford University Press, 1990, 154-188.

A forceful expression of the conviction that despite the result, due in part to the author himself, that the currently accepted axioms can neither prove nor refute Cantor's Continuum Hypothesis, it is nonetheless either true or false, and more likely the latter.

Holmes, M. Randall, 'Alternative Axiomatic Set Theories,' The Stanford Encyclopedia of Philosophy [http://plato.stanford.edu/archives/fall2014/entries/settheory-alternative/]*.

A survey of heterodox axiomatic set theories, which are many and varied, though none has a large following.

Hrbacek, Karel and Thomas Jech. Introduction to Set Theory, 3rd ed. New York: Marcel Dekker, 1999.

A much-used modern introductory textbook, intended for undergraduate mathematics courses but usable for independent study, with extensive problems.

Jech, Thomas. Set Theory, 3rd ed., Berlin: Springer, 2003.

A comprehensive graduate-level textbook, covering results of Cohen, Gödel, Jensen, Martin, Shelah, Silver, Solovay, Steel, Woodin, and others.

Kanamori, Akihiro. 'Introduction.' In Handbook of Set Theory, vol. 1. Edited by Matthew Foreman and Akihiro Kanamori. Berlin: Springer, 2010, 1-92.

An overview of the whole field as it appears to leading figures today, prefacing a multivolume compendium directed at future researchers.

Kechris, Alexander. Classical Descriptive Set Theory, Berlin: Springer, 1995.

A textbook on the side of set theory, called 'descriptive' as contrasted with 'combinatorial', closest to the subject's original home in mathematical analysis.

Zermelo, Ernst. Collected Works - Gesammelte Werke, vol. 1,, ed. Heinz-Dieter Ebbinghaus, Craig G. Fraser, Akihiro Kanamori, Berlin: Springer, 2010.

German originals with facing English translations of the fundamental papers of the

founder of axiomatic set theory.

Proof theory

David Hilbert in the 1920s introduced a *metamathematics* or *theory of proof*, in which formal counterparts of the proofs used as the *method* of study everywhere in mathematics became the *objects* of study. His original aim was to find a consistency proof for modern, abstract, set-theoretic mathematics that might silence the intuitionists and other critics. For what became of this program see Zach 2015. In the wake of Gödel's work (on incompleteness results, now most readily available in Gödel 1986) it is recognized that this aim cannot be achieved in its original form, but rather that we everywhere face trade-offs between the power of axiom systems (their ability to answer mathematical questions) and their riskiness (the potential danger of collapsing in contradiction). The delicate interplay of power and risk has since been intensively investigated, especially in so-called reverse mathematics, as surveyed in Simpson 1985, which by 'proving axioms from theorems' attempts to the determine the least risky assumptions powerful enough to yield this or that classical mathematical result. Meanwhile, the tools developed in connection with Hilbert's program by Gerhard Gentzen and others have come to live a life of their own, and developed into the subject represented at the textbook level by Takeuti 1987, and at a more advanced level in Buss 1998, with a high-level survey in Feferman 2000. Especially important for philosophers have been so-called natural deduction proof procedures, introduced independently in Gentzen 1934/1935 and Jaśkowski 1934, which not only are favored in many introductory textbooks, but also have inspired the philosophical idea that the *meanings* of logical operators (connectives and quantifiers) are to be explained in terms of rules of proof rather than conditions of truth, an idea that has in turn been made the basis for motivating arguments by proponents of intuitionistic logics.

Buss, Sam, ed. Handbook of Proof Theory, Amsterdam: Elsevier Science B.V., 1998.
A collection of high-level surveys by leading contributors of the many and varied parts of a sprawling field, including connections with computing.

Feferman, Solomon. 'Highlights of Proof Theory.' In Proof Theory: History and Philosophical Significance. Edited by V. F. Hendricks, S. A. Pedersen, K. F. Kørgensen, Dordrecht: Kluwer Academic Publishers, 2000, 11-34.
An historical survey of the main lines of development of proof theory from Hilbert on, emphasizing the increasing role of infinitistic methods.

Gentzen, Gerhard, 'Untersüchungen über das logsiche Schliessen.' *Mathematische Zeitschrift* 39 (1934/1935): 176-210 and 405-431. Translated as 'Investigations into Logical Deduction.' In M. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, Amsterdam: North-Holland, 1969: 68-131.
Pioneering work both on the so-called *sequent calculus* and on natural deduction.

Gödel, Kurt. 'Über formal unentscheidbare Sätze der Principia mathematical und verandter I' and 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems I.' In Kurt Gödel Collected Works Volume I, Publications 1929-1936. Edited by Solomon Feferman et al., Oxford: Oxford University Press, 1986.
An article widely viewed as the single most important publication ever in mathematical logic, presented in the German original with facing English translation by van Heijenoort.

Jaśkowski, Stanisław, ‘On the Rules of Suppositions in Formal Logic.’ *Studia Logica* 1 (1934): 5-32. Reprinted in S. McCall, editor, *Polish Logic 1920-1939*. Oxford: Oxford University Press, 1967: 232-258.

An account of version of natural deduction entirely independent of Gentzen’s work, representing a case of nearly simultaneous discovery of a major idea.

Simpson, Stephen G. ‘Friedman’s Research on Subsystems of Second Order Arithmetic’. In Harvey Friedman’s Research on the Foundations of Mathematics. Edited by L. A. Harrington, M. D. Morley, A. Scedrov, S. G. Simpson, Amsterdam: North Holland, 1985.

Remains perhaps the best short introduction to the aims and claims of Harvey Friedman’s reverse mathematics.

Takeuti, Gaisi. Proof Theory, 2nd ed., Amsterdam: North Holland, 1987.

A classic textbook of main-line proof theory, focused on so-called sequent calculus and cut-elimination, deriving from Gentzen, with coverage broadened in appendices (by Georg Kreisel, Wolfram Pohlers, Stephen G. Simpson, Solomon Feferman).

Zach, Richard. ‘Hilbert’s Program,’ The Stanford Encyclopedia of Philosophy [http://plato.stanford.edu/archives/sum2015/entries/hilbert-program/]*.

A concise historical and philosophical account of Hilbert’s program, the damage done it by Gödel’s incompleteness theorems, and what nonetheless survives from it, all in the light of recent research.

Recursion theory

For classical sentential logic there are decision procedures, such as the method of truth tables taught in introductory text books, that in principle will always tell one in a finite amount of time whether a given argument form is valid. For classical predicate logic there are no such decision procedures: There are proof procedures that always, if a given argument form is valid, will always tell one that it is; but there are no disproof procedures that always, if a given argument form is not valid, will tell one that it is not. The rigorous statement and proof of such results requires a rigorous definition of ‘effective decidability’ and ‘effective computability’ such as emerged in the form of ‘Church-Turing thesis’ from the work of Alonzo Church and Alan Turing (he of the famous machines) in the 1930s. The whole subject of theoretical computer science eventually emerged from these studies, a process described in Davis 2011. These developments, and the notion of Turing machine in particular, have had considerable influence on philosophy of mind as well as philosophy of mathematics. Key papers are made available in Davis 2004. Textbook accounts at an elementary and more advanced levels are to be found in Boolos et al. 2007 and Cooper 2004, and advanced surveys in Griffor 1999. What became of the original motivating problems for the field can be seen from Börger et al. 1997 and Matiyasevich 1993.

Boolos, George S. and John P. Burgess and Richard C. Jeffrey, Computability and Logic, 5th ed., Cambridge: Cambridge University Press, 2007.

An undergraduate-level textbook of intermediate-level logic, covering standard material through the Gödel theorems, with an account (mainly by Jeffrey) of the equivalence of various notions of computability, including Kleene’s in terms of recursive functions and Turing’s in terms of idealized machines.

Börger, Egon and Erich Grädel and Yuri Gurevich, The Classical Decision Problem, Berlin: Springer, 1997.

A modern account of the *Entscheidungsproblem*, or problem of determining whether a given logical formula is valid, proved undecidable by Church and by Turing, but having many decidable special cases, with connections to superficially different-seeming decidability questions such as the domino problem.

Cooper, S. B. Computability Theory, Boca Raton, Florida: Chapman & Hall/CRC, 2004.

A modern undergraduate-level textbook, covering all the standard topics.

Davis, Martin, ed. The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions, Mineola, New York: Dover, 2004.

A corrected reproduction of the 1965 original anthology of fundamental papers by Kurt Gödel, Alonzo Church, Alan Turing, and others.

Davis, Martin, The Universal Computer: The Road from Leibniz to Turing, New York: W. W. Norton & Co, 2011.

A semi-popular history of conceptual developments in mathematics and logic leading up to the creation of modern digital computing, requiring very little background on the part of the reader.

Griffor, Edward R., ed. Handbook of Computability Theory, Amsterdam: Elsevier Science B. V., 1999.

A collection of high-level surveys of those parts of the theory of computability that remain more in the domain of mathematical logic than than of computer science (not that there is a sharp division).

Matiyasevich, Yuri. Hilbert's Tenth Problem. Cambridge, Massachusetts: MIT Press, 1993.

An account of one of the most famous problems that has turned out to be effectively undecidable, by the mathematician who proved it to be so, showing the interplay of recursion theory and number theory.