Why Priest's reassurance is not reassuring*

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In the service of paraconsistent (indeed, 'dialetheic') theories, Graham Priest has long advanced a non-monotonic logic (viz., MiLP) as our 'universal logic' (at least for standard connectives), one that enjoys the familiar logic LP as its monotonic core [10, Chs 16, 19]. In this paper, I show that MiLP faces a dilemma: either it is (plainly) unsuitable as a universal logic or its role as a 'universal logic' (indeed, its role full stop) is a mystery. While familiarity with the basic ideas of dialetheism [4, 10] is assumed, formal details of the target logics are relegated to an appendix; the basic problem is evident without them.

1 *LP and MiLP: the basic picture*

The logic LP (for 'logic of paradox') offers many virtues in philosophy, most notably for underwriting theories subject to paradox (e.g., truth, properties, etc).[1] But LP is also a notoriously weak logic. Example: LP never sanctions the inference from $\{\neg A, A \vee B\}$ to B. This inference goes wrong where A is a glut (and, hence, $A \wedge \neg A$ true) and B untrue. LP guards against that threat.

But one might think that *for just that reason* a subtler, more liberal logic is required – one that sanctions the inference of q from $\{\neg p, p \vee q\}$ in cases where $\neg p$ is a regular old (non-glutty!) truth.[2] After all, even by the wildest dialetheic lights, gluts still remain relatively few and far between. And one might want a logic that reflects all this: a logic which, in effect, is classical when it can be – when consistency is an option – but otherwise behaves along LP lines. Towards this end, Graham Priest has long advanced his (non-monotonic) 'minimally inconsistent LP' (or MiLP) [9, 10].[3]

The point of MiLP, as Priest says, is to be 'the "universal" logic; LP is its monotonic fragment' [10, fn 36, p 275]. The basic picture is an 'adaptive' one [2, 3]: roughly, classical logic gets things right except when the premise set has no consistent way of being satisfied. An immediate effect is a more liberal relation of validity: $\{\neg p, p \vee q\}$ has a classical model; and any such model is one in which q is true (satisfied). Hence, details aside (see Appendix), our new 'universal' validity relation (viz., MiLP) sanctions the given inference.

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[1] This logic was first advanced for this purpose (i.e., indeed, for underwriting 'glutty' theories) by Asenjo under the name 'calculus for antinomies' [1], but later independently discovered and widely applied by Priest [8, 10] and others in the service of 'dialetheism' – the view that there are 'gluts' or 'true falsehoods', that is, true sentences of the form $A \wedge \neg A$.

[2] Throughout, p, q, and r (with or without subscripts) are atomics; A and B any sentences.

[3] See Appendix for a primer on the formal details; but the principal philosophical problem is evident without such details (which is why they're relegated to the Appendix).

But what about the case where we have that p is a glut – that $p \wedge \neg p$ is in fact true? In particular, what if, in addition to the truth of $\neg p$, we add that p is also true, resulting in the premise set $\{p, \neg p, p \vee q\}$? In *this* case, surely we want our logic *not* to deliver q (since the 'gluttiness' of p is sufficient to satisfy $p \vee q$ even if q is unsatisfied). The nice feature of MiLP is that it does *not* sanction the inference to q from the classically unsatisfiable set $\{p, \neg p, p \vee q\}$, and indeed counts it as invalid. MiLP is a non-monotonic logic built precisely to be sensitive to inconsistency that cannot logically be avoided.[4]

This sounds like the perfect world: our logic (viz., MiLP) reacts to inconsistency in the appropriate (LP-tolerant) fashion, but otherwise carries on with classical logic. But this is not a horses-for-courses plurality of logics; it's constructed to be one simple (non-monotonic) 'universal logic' which has the right sort of adaptive behavior – appropriately LP-like in responding to unavoidable inconsistency, but otherwise robustly stronger than LP in many respects. But therein lies a worry.

2 *Priest's reassurance*

There are simple proofs that many important theories (e.g., truth, properties, set theory, arithmetic, more) avoid triviality under LP [10]. But MiLP is stronger than LP; and MiLP, not LP, is to serve as the 'universal logic' over any such domain of inquiry. Its additional strength, combined with its intended role, raises a concern about MiLP.

To see the worry, let X be any theory (any set of sentences), and let X^{lp}, X^m and X^c be the LP-, MiLP- and CPL (classical) consequences of X, respectively. Priest frames the worry concerning MiLP as follows.

> [That MiLP is 'a more generous inference engine' than LP] raises the possibility that X^m might collapse into triviality when X^{lp} does not. This would obviously be unfortunate, since it would show that there are perfectly sensible (non-trivial) contexts where [MiLP] could not be used. Its theoretical legitimacy would therefore have to be restricted, just as that of classical logic is. It would be very reassuring, therefore, if, whenever X^{lp} is non-trivial, so is X^m. Let us therefore call this property *Reassurance*. [10, p. 226]

And Priest [10, Ch. 16] gives just such reassurance: he proves that, for any theory (i.e., set of sentences) X, if X^{lp} is non-trivial then X^m is non-trivial too.[5] As mentioned above, we have proofs that LP avoids trivializing many of our important theories; and Priest's reassurance therefore ensures that MiLP – though sanctioning many more inferences – avoids trivializing them too. Such reassurance is essential given that, as Priest suggests, MiLP is to serve as our all-purpose logic. But is Priest's reassurance enough reassurance?

[4] A logic is non-monotonic if adding to the premise set can go from a valid argument to an invalid one. Like many non-monotonic logics, MiLP is *not* closed under uniform substitution (of non-logical vocabulary). Example: where $\vdash_m$ is MiLP consequence (see Appendix), we have $\{\neg p, p \vee q\} \vdash_m q$ but $\{p, \neg p, p \vee q\} \nvdash_m q$. (Notation: throughout, $\vdash_{lp}$, $\vdash_m$ and $\vdash_c$ are LP-, MiLP- and CPL-validity relations, respectively (where CPL is classical propositional logic).)

[5] This is proved for any X in the propositional fragment. Certain restrictions, irrelevant to present concerns, are imposed on the first-order case [10, Ch. 16].

3 *Why the reassurance is not reassuring*

Triviality is but the limiting case of theoretical badness. The trivial theory contains all sentences; and while it thereby contains all truths, it also contains every *untruth* – from the absurd ones to the sensible but contingently untrue. And while containing *all* untruths is bad, so too is containing *any* untruths.

Call a theory X *true* iff it contains no untrue sentences; and call X *untrue* iff it contains some untrue sentence.[6] Suppose, now, that X^{lp} is true (and, hence, non-trivial). What we expect from our would-be universal logic MiLP is:

- *General Reassurance.* If X^{lp} is true, then X^m is true.

Priest's reassurance delivers the limiting case: if X^{lp} is *non-trivial*, then so is X^m. But surely we expect at least general reassurance from a would-be universal logic. The trouble: Priest's MiLP fails to satisfy general reassurance.

A useful way to see the problem is to observe that MiLP, like CPL but unlike LP, delivers r from $p! \vee r$, where, for ease of notation, $p!$ is $(p \wedge \neg p)$.[7]

- $p! \vee r \vdash_m r$. (Proof: Appendix §A.3.)

But now the problem is plain.[8] Let r be any untruth; and let p be a glut, so that $p!$ is true, and hence, by LP semantics, $p! \vee r$ is true. Let $X = \{p! \vee r\}$ be our (true) theory. Then X^{lp} is a true theory (proof: exercise); however, X^m is *untrue*, since r is in X^m, since $X \vdash_m r$. Hence, general reassurance is refuted.

Of course, just as MiLP satisfies the non-triviality limit of general reassurance (viz., Priest's reassurance), so too it satisfies the other limit, namely, *full truth*: if X^{lp} contains *all* truths (and no untruths), then so too does X^m.[9] But is this sense of general reassurance enough reassurance for our would-be universal logic MiLP? No; it makes a mystery of MiLP's role qua universal logic – indeed, its role full stop. If we have to wait until our theory contains *all* truths before safely applying MiLP, then we don't have to wait for MiLP at all.

4 *From pre-theory to final theory via MiLP?*

As the foregoing makes plain, whatever sense there may be to MiLP's being a 'universal logic', it isn't in the sense of being appropriately reliable – in particular, truth-preserving – over all domains or theories (shy of full-truth); MiLP isn't, after all, a *closure operator* on theories (e.g., not being monotonic or, for that matter, transitive). But perhaps MiLP is to be thought of as 'universal' in another sense: it's to be fed our (say) 'pre-final theory' and, in turn, generate our final theory by advising what to add. Let X^{lp} be our penultimate theory – our 'nearly final theory', as it were. (In other words, think of X as a late-stage theory of inquiry, and then our penultimate or 'pre-final' theory X^{lp} is the result of closing X under LP.) In the face of LP's weakness, we treat our given

[6] This treats the empty theory $\emptyset$ as true by omission: it avoids being untrue. Throughout, my use of the term 'untrue' is formally modeled (see Appendix §A.1) as having value 0.

[7] The abbreviation of $A \wedge \neg A$ by $A!$ is Priest's notation [10, Ch. 16].

[8] Priest [10, Ch. 16] talks about a related – though more complicated – case, but he seems to overlook the current problem.

[9] Thanks to an anonymous *Analysis* referee for prompting discussion of this point – and for prompting §4–§5 more generally. (Priest himself won't balk at the idea of there being a *set* X of all truths; I ignore such potential concerns here.)

pre-final theory X^{lp} as only *nearly final* – waiting for something to take X^{lp} and, in turn, deliver our final theory. And it's here, perhaps, where MiLP is supposed to play a role: it serves as 'universal' in the sense of taking our nearly final theory X^{lp}, *as a whole*, and wringing out the remaining consequences to appear in our final theory, consequences that LP is too weak to deliver.

Does this way of looking at MiLP – namely, as the last step from our penultimate theory to our final theory – avoid the problem? The answer is *no*, and the problem the same. Let X^{lp} be our late-stage pre-MiLP theory – our LP-closed 'nearly final theory', waiting only for us to apply MiLP to squeeze out our final theory. Suppose that $\{q! \vee r\} \subset X^{lp}$ but that neither $q!$ nor r is in the theory. The problem is now evident. We're to use MiLP to squeeze out our final theory from pre-theory X^{lp}, adding whatever, according to MiLP, follows from X^{lp}. Since $q! \vee r$ is in X^{lp} and $q!$ not in X^{lp}, MiLP tells us to add r to our overall theory. But r might be untrue – indeed, may well be absurd (e.g., 'Priest is a scrambled egg'). The original problem remains.

Lest one think that 'in real life' we won't get this problem, consider a simple example: we are convinced, by the powerful parade of philosophical or scientific discovery, that *some apparent absurdity is true*, but we remain unsure about which one – unsure about the *witness* for our existential claim or, simplifying, disjunction. (A common analogous case: theists convince us that *some god or other exists*, but we remain unconvinced as to which one. Similarly, we might be convinced that *something outrageous that Richard Routley said is true* though unsure of the witness.) Now, among the apparent absurdities are not only contradictions but non-contradictory absurdities (e.g., 'Priest is a scrambled egg'). In the simplest such case, our theory contains some disjunction of gluts and non-glutty absurdities without containing any witness for the disjunction.[10] But this is enough for the problem: MiLP delivers r from the given disjunction, even where r is untrue. The original problem remains.

5 *MiLP restricted to* prime *theories?*

As a way to ensure its safety, one might restrict MiLP to so-called *prime theories*, where a theory X is *prime* iff a disjunction is in X only if one of the disjuncts is in X. If primeness, so understood, is imposed on our pre-MiLP theory X^{lp}, then the foregoing problem is avoided: primeness requires that either r or $q!$ be in our pre-MiLP theory if, as the problem assumes, $q! \vee r$ is in our pre-MiLP theory. Either way, the problem above seems to be no problem: given primeness, our pre-MiLP theory includes either $\{r, q! \vee r\}$ or $\{q!, q! \vee r\}$. In the former case, MiLP's advice to 'add' r is superfluous, since we've got it anyway; and in the latter case, MiLP won't deliver r, since $\{q!, q! \vee r\} \nvdash_m r$.

But does this help secure a role for MiLP – restricting MiLP's range of application to prime theories? No; it makes a mystery of MiLP's role.

To see the point, let a *complete* theory be a *negation-complete* theory: for all A, either A is in the theory or its negation $\neg A$ is in the theory. What is important to observe is that any theory that is closed under excluded middle and also prime is complete.[11] But excluded middle is valid in LP; and so for

[10] The point applies in the first-order case with existential claims; but the simpler propositional case is sufficient for present purposes.

[11] Closing X under excluded middle puts $A \vee \neg A$ in the resulting theory for all A. Hence,

any theory X the LP-closed theory X^{lp} contains $A \vee \neg A$ for all A. Hence, if we demand that our theory X^{lp} be *prime* before we can safely apply MiLP, we thereby demand that X^{lp} be complete. And this demands too much. What further consequences do we need MiLP to draw from our *complete* theory? None. Invoking primeness in an effort to secure a safe role for MiLP takes us back to the full-truth problem (see §3): the apparent superfluity of MiLP.

6 *The upshot*

Priest suggests that MiLP be seen as our 'universal logic' which enjoys LP as its weaker (monotonic) base. Because MiLP can deliver more consequences from a theory than LP does, Priest recognizes the need for reassurance. But what Priest's 'reassurance' gives us is only that our theories are safe from the limiting case of theoretical badness – triviality. This is not reassuring enough if MiLP is indeed to be seen as our would-be universal logic. Indeed, what is clear is that MiLP is inadequate (indeed, precisely not reassuring) as a general logic: it can take us from true theories to untrue ones. This problem is avoided by restricting MiLP to prime theories; but this, in turn, makes a mystery of MiLP's role – full stop. In the end, it appears that MiLP is either inadequate as a would-be universal logic or unnecessary.[12]

Nothing in what I've argued undermines the virtues of LP, but only Priest's strategy of overcoming the apparent weaknesses of LP via MiLP. My own view is that the apparent weaknesses of LP are overcome via an alternative route [5]; but I leave this for a larger project [6].

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Postscript note. After this paper was completed, Aaron Cotnoir pointed me to very recent work that raises even more questions about the viability of MiLP, namely, the work of Marcel Crabbé [7]. In light of the current paper and Crabbé's work, it seems to me that searching for an alternative route towards overcoming LP's apparent deficiencies is well-motivated [5].

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if primeness is further imposed, either A or $\neg A$ is in the resulting theory, for all A.

[12]Another tension arises for MiLP advocates, though I only mention it here. The example in §3 shows that MiLP is a *non-transitive* logic:

- $q! \vdash_m q! \vee r$. (Proof: this holds in LP.)
- $q! \vee r \vdash_m r$. (Proof: see Appendix §A.3.)
- $q! \nvdash_m r$. (Countermodel: any $v \in \mathbb{V}$ such that $v(q) = 0.5$ and $v(r) = 0$.)

The tension: if you're going to go with a non-transitive logic for which uniform substitution fails (as Priest does with MiLP), why not instead go with Ripley's much stronger such logic ST [11, 12], wherein disjunctive syllogism – and more – holds? My own view is that we should keep logic's transitivity and uniform-substitution; but if these are jettisoned to overcome weaknesses of LP, why not go the stronger Ripley route? But I leave this for debate among the given non-transitive (non-uniform-substitution, etc.) advocates.

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A *Appendix: LP and MiLP*

This is a quick primer on MiLP.[13] Since the model theory (or 'semantics') of MiLP presupposes that of LP, I first present LP, and then present MiLP.

[13] Even though the philosophical applications of the logic arise from the first-order level, my chief concern in this paper can be raised at the simpler boolean-connectives level, and so I restrict focus to the boolean connectives. (Notation: Priest uses 'LPm' for MiLP, but the appended 'm' can sometimes get in the way when the need for subscripts arise (not here), and so I generally prefer to use 'MiLP' (as I have here).)

A.1 *LP validity*

The logic LP is a sublogic of classical logic (CPL). In effect, LP agrees with CPL on many fronts but recognizes a third option for sentences: namely, the 'glutty' option – being true and false (i.e., true and the negation is true too). This third option counts as 'a way of being true'. Hence, we make two changes to the CPL model theory: we expand our set of classical semantic values to a 3-membered set $V = \{1, .5, 0\}$, and we *designate* the middle value so that a sentence A counts as being *satisfied* (or, if you like, 'true') just if it has the value 1 or 0.5. Of course, none of the sentences will be satisfied (or true) *simpliciter*, but rather only relative to an interpretation (or, as I'll say, a valuation). And here we simply stick with the CPL setup except for changes brought on by the third value. In particular, we interpret the language via (total) functions $v : S \longrightarrow V$ from the set S of sentences into V. Such functions are called *valuations*; and the set $\mathbb{V}$ of *admissible valuations* are all and only those valuations that obey the following (familiar) constraints:

- Negation: $v(\neg A) = 1 - v(A)$.
- Disjunction: $v(A \vee B) = \max\{v(A), v(B)\}$.
- Conjunction: $v(A \wedge B) = \min\{v(A), v(B)\}$.

Worth observing is that these are precisely the familiar classical conditions.

Towards defining validity, let v be in $\mathbb{V}$, and let A be any sentence, and let X be any set of sentences. Then v *satisfies* A iff $v(A) \in \{1, .5\}$, and satisfies X iff v satisfies everything in X. Then LP validity is defined along standard lines:

- $X \vdash_{lp} A$ iff there's no $v \in \mathbb{V}$ that satisfies X but doesn't satisfy A.

Given all this, it is clear that classical validity $\vdash_c$ is an extension of $\vdash_{lp}$, that is – here conceiving of the validity relations extensionally as pairs $\langle X, A\rangle$ of arguments – the latter is a subset of the former. Hence, anything LP-valid is CPL-valid. But the relation is proper inclusion: we have both $\neg A, A \vee B \vdash_c B$ and $A \wedge \neg A \vdash_c B$ in the classical case, but $\neg A, A \vee B \nvdash_{lp} B$ and $A \wedge \neg A \nvdash_{lp} B$ in the weaker (paraconsistent) LP case. A counterexample in both cases is any $v \in \mathbb{V}$ such that $v(A) = 0.5$ but $v(B) = 0$.

A.2 *MiLP validity*

MiLP is built to be a universal 'adaptive' logic that preserves the virtues of LP (viz., responds well to inconsistency) but sanctions classical inferences where possible – namely, where the premise set (or, generally, theory) is logically consistent or, in short, negation-consistent. To make this precise we appeal to the relation between CPL models and LP models, as follows [9, 10].

For any sentence A, we let $A!$ abbreviate $A \wedge \neg A$. In LP, we have it that $A!$ is satisfied by $v \in \mathbb{V}$ iff $v(A) = 0.5$. (The proof falls out of the clauses on negations and conjunctions.) With this in mind, take any $v \in \mathbb{V}$ and define $v!$ to be the 'inconsistency chunk' of v, namely,

$$v! = \{x : x \text{ is an atomic sentence and } v(x) = 0.5\}$$

Because of the truth-functionality (or, generally, value-functionality) of LP's model theory, we have it that if $v \in \mathbb{V}$ assigns 0.5 to *all* atomics, then $v!$ is the 'trivial model' (i.e., assigns 0.5 to all sentences). Dually, if $v \in \mathbb{V}$ is a *classical* valuation (recall that $\mathbb{V}$ is a superset of CPL's admissible valuations), then $v!$ is the anti-trivial valuation, namely, $v! = \emptyset$. Accordingly, $v!$ serves as a convenient 'measure of inconsistency' on a valuation $v \in \mathbb{V}$, and we can use this to define a (strict) partial ordering $\prec$ on valuations in terms of strict inclusion:

$$v_1 \prec v_2 \text{ iff } v_1! \subset v_2!$$

And with this ordering we get to the important idea: namely, *minimally inconsistent models*.

- We say that $v \in \mathbb{V}$ is a *model of* X iff v satisfies X.
- Let $v \in \mathbb{V}$. Then v is a *minimally inconsistent (mi-) model of* X iff v is a model of X and, for any $v' \in \mathbb{V}$, if $v' \prec v$ then v' does not model X.

Worth noting is that if X has a CPL model, then no LP model that assigns 0.5 to anything in X is an mi-model of X.

MiLP validity is defined in terms of such models (*mi-models*, for short):

- $X \vdash_m A$ iff there's no mi-model of X that fails to be a model of A.

As remarked above, MiLP is *non-monotonic*. An important example:

$$\neg p, p \vee q \vdash_m q$$

but

$$p, \neg p, p \vee q \nvdash_m q$$

MiLP is (supposed to be) built to give you the right consequences from a theory (i.e., set of sentences). Consider the two cases above. In the latter case, MiLP sees that there's no logical path towards consistently satisfying the premise set, and so MiLP throws the matter 'down' to LP to generate its consequences. In the former case, where the given set is classically satisfiable, MiLP happily follows its stronger CPL side to generate consequences that LP fails to deliver.

In fact, the relationship between CPL, LP, and MiLP is closer than my metaphorical talk suggests. The following facts obtain [10, Ch. 16].

- $X^{lp} \subseteq X^m \subseteq X^c$. (As Priest notes, these are generally *proper* inclusions.)
- Suppose X^c is non-trivial, that is, that X^c doesn't contain all sentences. Then $X^m = X^c$. (Proof: exercise.)

The second fact is revealing. The idea, figuratively, is that unless CPL trivializes a theory X, then MiLP looks to CPL to deliver the consequences of X.

A.3 *Proofs of §3 facts*

We can now see why the critical fact(s) invoked in §3 hold:

- $p! \vee r \vdash_m r$. *Proof:* there are CPL models of $\{p! \vee r\}$, and each is a model of r. Hence every mi-model of $\{p! \vee r\}$ is a model of r.[14]
- $p! \vee r \nvdash_{lp} r$. *Countermodel:* any $v \in \mathbb{V}$ such that $v(p) = 0.5$ and $v(r) = 0$.

[14] A different proof relies on the second noted fact above: that if X^c is non-trivial, then $X^c = X^m$. Hence, since $\{p! \vee r\}^c$ is non-trivial, said fact delivers that $\{p! \vee r\}^c = \{p! \vee r\}^m$. Consequently, since r is in $\{p! \vee r\}^c$, so too with $\{p! \vee r\}^m$.