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A NOTE ON FDE 'ALL THE WAY UP'

ABSTRACT. There is a natural 'combinatorial argument' for the philosophical view that if the standard (so-called classical) account of logical consequence (henceforth, logic) is right about logic's fundamental truth values (viz., The True and The False), then FDE, a well-known subclassical logic [1, 2, 6, 8], is a more natural account of the space of 'logical possibility'. Without officially endorsing that argument, our aim in this note is to defend it from an otherwise powerful objection. Our defense rests on an explicit generalization of a result by Priest [11]. In particular, by way of answering the target objection, we explicitly show that after combining the standard (classical) values in $\{\top, \bot\}$ to get a space of four values, as FDE demands, the given process of combining values 'all the way up' to α many values, for *any* ordinal α, results in the same account of logical consequence (viz., FDE).

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1. Introduction: philosophical background

Logical consequence can be thought of as determining the broadest space of possibilities – the logical possibilities. Theories themselves may (and often do) constrain the space of theoretical possibilities – treating many of the logical possibilities as theoretical impossibilities. Logical vocabulary, on this picture, is the sparse set of vocabulary common to all of our true theories.

On the standard (so-called classical-logic) account of logical consequence, the space of possibilities for any sentence of any true theory requires that the sentence either be true according to the possibility or false according to the possibility, and never both true and false in the possibility. Modeling logical possibilities as familiar first-order models, one can think of the classical constraint on sentences as being reflected in the set of 'fundamental truth values', namely, $V = \{\top, \bot\}$, and the restriction that every sentence be mapped to exactly one such fundamental value.

Our aim here is not to elaborate or advance or defend such a view of logical consequence and its role in true theories, though we find the view to be both philosophically viable and plausible. In this note we focus on a specific *subclassical* account of logical consequence, one whose space of logical possibilities is broader than the standard classical-logic space. The specific subclassical account of consequence on which we focus is often called 'FDE' for 'first-degree entailment', an account of the (standard) first-order (extensional, 'truth-functional') vocabulary shy of identity. This account of logical consequence was vigorously explored in the name of so-called relevance logic (sometimes *relevant logic*) by Belnap [1, 2, 6] and Dunn [8] and many others, and has been applied in philosophy in various directions [3, 4].

Our concern in this note is with a natural 'combinatorial' argument sometimes advanced in favor of FDE over its narrower (and stronger) cousin, namely, classical logic. The argument turns on a very natural idea about logic's liberal recognition of possibilities 'around The True and The False' – around logic's fundamental truth values. The argument, rarely (if ever before now) in print, runs as follows.

(1) Logic's fundamental semantic values are The True and The False: $V = \{\top, \bot\}$.[1]
(2) But the two fundamental values (viz., The True and The False) afford four possibilities, namely:
   (a) Sentence A has *just* the fundamental value $\top$.
   (b) Sentence A has *just* the fundamental value $\bot$.
   (c) Sentence A has *neither* fundamental value.
   (d) Sentence A has *both* fundamental values.

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[1]And the fundamentality of these categories is reflected in the standard account of first-order – indeed, even standard propositional – vocabulary. In particular, logic has two fundamental unary sentential connectives (or operators); one is logic's falsity operator 'it is false that...' (viz., logical negation) and the other, even though nearly always omitted from explicit lists because it's logically redundant, is logic's truth operator 'it is true that...' (viz., logical nullation, or the 'null operator', as Anderson and Belnap call it [1, 2]).

(3) Since logic itself determines the broadest space of possibilities, the combinatorial options in (2) are logical possibilities (given, again, that the space of logical possibilities is supposed to be the broadest space of possibilities).

(4) Hence, the space of logical possibilities allows for four 'semantic statuses' – four semantic categories that a sentence may inhabit – arising from the set of logic's fundamental truth values; and these statuses or categories for any sentence can be represented by a four-element set $\{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}\}$, where the elements are heuristically named to reflect the given status:
   (a) Just True: t.
   (b) Just False: f.
   (c) Gappy (*neither* True nor False): n.
   (d) Glutty (*both* True and False): b.

We do not (in this paper) endorse the foregoing argument for FDE over the standard account. Our aim is rather to defend the argument against an immediate and very natural objection. The *objection*:

(5) If the argument from (1)-(4) supports FDE over the narrower classical account, then it had better support FDE for *any* possible combination of the space of 'semantic statuses'.

(6) One possible space of semantic statuses goes from $V_0 = \{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}\}$ to $\wp(V_0)\backslash\{\emptyset\}$, representing, in turn, every (positive, non-null) combination of V_0's elements to get an even broader space of 'semantic statuses';[2] and then goes from that set to every (positive) combination of its elements; and so on, into the transfinite for any ordinal α.

(7) Hence, the argument from (1)-(4) supports FDE only if FDE is the consequence relation determined by the full space of (positive) combinations for any ordinal.

Our (modest) aim here: we show (in §7.2), by extending a result of Priest [9], that FDE *is* the consequence relation determined by the full space of α many positive combinations of V_0 for any ordinal α. Hence, the philosophical objection to the natural 'combinatorial argument' for FDE is met.[3]

2. Many-valued semantics

We use the usual definition of a many-valued logic and (propositional) language [5, 10, 12],[4] where, for (propositional) language/syntax $\mathcal{L}$, a structure $M = \langle V, D, \delta\rangle$ gives the elements of $\mathcal{L}$'s semantics, where V contains values, $D \subseteq V$ the designated values (in terms of which $\mathcal{L}$'s consequence relation is defined), and δ maps each n-ary connective $\odot$ to its corresponding function $\delta_\odot$ from V^n into V.

For any such semantics M, the ingredients in M are connected to the syntax of $\mathcal{L}$ by an 'interpretation' – a *valuation* – μ that takes $\mathcal{L}$'s sentences (atomics and in turn molecular) to elements of V via the usual 'compositional constraint' constraint: namely, $\mu(\odot(A_1, \ldots, A_n)) = \delta_\odot(\mu(A_1), \ldots, \mu(A_n))$, where $\odot$ is an n-ary connective of $\mathcal{L}$ and $\delta_\odot$ its n-ary operator in M.

The (set-sentence) consequence relation for $\mathcal{L}$ is defined standardly in the absence-of-counterexample way, where a counterexample to $\Sigma \therefore A$ is an interpretation (per above) wherein everything in Σ is designated but A is not designated.

3. Homomorphisms on Semantics

Given two semantics $M^* = \langle V^*, D^*, \delta^*\rangle$ and $M = \langle V, D, \delta\rangle$ on the same language $\mathcal{L}$, a homomorphism from M^* to M is a function $\theta : V^* \to V$ such that $x \in D^*$ iff $\theta(x) \in D$ and, additionally, for any n-place

[2] The 'null combination' of a set of values just is the set of values; the target argument points only to positive or non-null combinations of semantic statuses.

[3] History of this note: Beall presented arguments for FDE as 'the one true logic' (understood a certain way) at his plenary talk at the 2018 North American ASL conference in Macomb IL USA. One of the arguments Beall advanced was a variant of the given 'combinatorial argument' above. Camrud, during the discussion period, immediately objected that if the given argument is a good argument then so too is an argument that extends the space of categories into the transfinite. Camrud's objection is similar to the one by Robert K. Meyer that motivated Priest's paper 'Hypercontradictions' [9], which is partly generalized by Priest in [11]. Our results in this note generalize Priest's results, but focus attention on answering the given objection to the given 'combinatorial argument' for FDE.

[4] Extending to the full first-order language should raise few difficulties but, as in his earlier results [11], we follow Priest in sticking to a propositional language.

connective $\odot$ in $\mathcal{L}$, with $\delta_\odot$ its corresponding function, $\theta(\delta^*_\odot(x_1, ..., x_n)) = \delta_\odot(\theta(x_1), ..., \theta(x_n))$. Note that for each valuation $v^*(\cdot)$ of M^*, the corresponding function $v(\cdot) = \theta(v^*(\cdot))$ is a valuation for M.

Theorem 1. *(Czelakowski [7], esp. Prop 0.3.3 in a fn., which Priest explicitly cites; and Priest [11]) Let $M^* = \langle V^*, D^*, \delta^* \rangle$ and $M = \langle V, D, \delta \rangle$ be semantics on the same language $\mathcal{L}$, with $\vDash_*$ and $\vDash$ defined in the usual way for all valuations $v^*(\cdot)$ of M^*. If $\exists\theta : V^* \to V$ s.t. θ is an onto homomorphism, then*

$$\Sigma \vDash_* A \textit{ iff } \Sigma \vDash A$$

Proof: For the forward direction, suppose $\Sigma \nvDash A$. Then for some valuation $v(\cdot)$, $v(A) \notin D$ but $v(B) \in D$ for every $B \in \Sigma$. By the axiom of choice, since θ is onto, we may choose θ^{-1} as a map from V to V^* s.t. if $x \in V$, $\theta^{-1}(x) = x^*$ for some $x^* \in V^*$ s.t. $\theta(x^*) = x$. Then $v^*(\theta(\cdot))$ is a valuation on M^* and $v^*(\theta(A)) \notin D$ but $v^*(\theta(B)) \in D$, for every $B \in \Sigma$, so $\Sigma \nvDash_* A$.

For the reverse direction, suppose $\Sigma \nvDash_* A$. Then for some $v^*(\cdot)$, $v^*(A) \notin D^*$ but $v^*(B) \in D^*$ for every $B \in \Sigma$. But then for $v(\cdot) = \theta(v^*(\cdot))$ we have that $v(A) \notin D$ and $v(B) \in D$ for every $B \in \Sigma$, so $\Sigma \nvDash A$. ■

4. Plurivalent extensions of logic

Using Priest's terminology [11, p.1], a semantics of a language is *univalent* when the 'connection' between the semantics and the syntax is achieved, per §2, via a function, assigning to each sentence of the language exactly one element in the space V of values. An interpretation is *plurivalent*, on Priest's usage (which we are adopting for the present discussion), just if the given 'connection' is a one-many ('positive') relation $\triangleright$ from all atomic sentences (propositional parameters) of the language into V,[5] and extended to all sentences (viz., molecular) via the *plurivalent compositional constraint* that

$$\odot(A_1, \ldots, A_n) \triangleright v \text{ iff } \exists v_1, \ldots, v_n(A_1 \triangleright v_1, \ldots, A_n \triangleright v_n \text{ and } v = \delta_\odot(v_1, \ldots, v_n))$$

4.1. **Plurivalent counterexamples and consequence.**

Importantly, where $\vDash_u$ is the consequence relation for a univalent interpretation of language $\mathcal{L}$, defined per §2 as an absence-of-counterexample relation, the corresponding plurivalent extension of $\vDash_p$ is likewise defined; but in the latter case a counterexample to $\Sigma \therefore A$ is an interpretation that assigns *at least one designated value* (i.e., value from $D \subseteq V$ in the semantics M) to each sentence in Σ while assigning no such (designated) value to A.

5. A Plurivalent Extension of FDE

Let $V_0 = \{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}\}$, the set of FDE truth values. The standard four-valued semantics of FDE interprets the connectives thus:

x	$\neg_0 x$
t	f
b	b
n	n
f	t

$x \wedge_0 y$	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

$x \vee_0 y$	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

where $\neg_0$, $\wedge_0$, and $\vee_0$ are the corresponding 'truth functions' to the 1- and 2-place connectives $\neg$, $\wedge$, and $\vee$.[6]

Plurivalent FDE allows sentences to take on any truth value in $V_1 = \wp(V_0) \setminus \{\emptyset\}$, where, as in §1, the elements of $\wp(V_0)\setminus\{\emptyset\}$ represent the possible 'positive combinations' of semantic values from V_0. Accordingly, plurivalent FDE is achieved by extending the standard truth functions (above) to plurivalent correlates:

$$\neg_1 X = \{\neg_0 x \,|\, x \in X\}$$
$$X \wedge_1 Y = \{x \wedge_0 y \,|\, x \in X \text{ and } y \in Y\}$$
$$X \vee_1 Y = \{x \vee_0 y \,|\, x \in X \text{ and } y \in Y\}$$

[5] A one-many relation from sentences to a set of values is said to be *positive* (in this context) just when each sentence is related to *something* by the relation. (As in §1 the null set would represent a null combination of prior statuses to which a sentence is related; but the driving argument concerns positive combinations of prior statuses, etc.) Our focus throughout is on 'positive' relations, so understood. But see the appendix, which generalizes a result of Priest's that covers even the non-positive one-many relations.

[6] For aesthetic reasons – though acknowledging potential confusion without context – we use this notation instead of Priest's '$\delta_\odot$' notation [11]; otherwise, we would soon be forced to use multiple subscripts.

Note that because FDE, as an account of logical consequence, is (semantically or model-theoretically) defined in terms of the interpretations it gives to the connectives and in terms of its absence-of-counterexample definition of consequence, both of these features must be preserved in the plurivalent extension of (univalent) FDE.[7] And that's what the given (above) plurivalent interpretations of the connectives do: they reflect the same so-called truth/falsity conditions that connectives have in FDE, where in the standard (univalent) FDE setting a sentence is said to be true on a valuation iff it takes the value t or b; it's false on a valuation (nb: different from being undesignated) if it takes the value f or b; and it's neither true nor false otherwise. (Accordingly, a conjunction $A \wedge B$ is true on a valuation iff both conjuncts are true on the valuation; it's false on a valuation iff at least one conjunct is false on a valuation; and it's neither true nor false otherwise. And so on for the other connectives.) The plurivalent FDE interpretations of the connectives preserve the given truth/falsity conditions definitive of FDE interpretations of the connectives.

5.1. **Plurivalent FDE valuations.** Plurivalent valuations are as expected: for $i = 0$ and $i = 1$ let v_i be a map from the propositional atoms (connective-free sentences) of a propositional language to V_i, and extend that map to a map of all sentences of the language by the respective plurivalent interpretations of the connectives, all in keeping with the plurivalent compositional constraint in §4.

5.2. **Designated values.** Let $D_0 = \{\mathsf{t}, \mathsf{b}\}$. Let D_1 be as follows:

$$D_1 = \Big\{\{\mathsf{t}\}, \{\mathsf{b}\}, \{\mathsf{t}, \mathsf{b}\}, \{\mathsf{t}, \mathsf{n}\}, \{\mathsf{t}, \mathsf{f}\}, \{\mathsf{b}, \mathsf{n}\}, \{\mathsf{b}, \mathsf{f}\}, \{\mathsf{t}, \mathsf{b}, \mathsf{n}\}, \{\mathsf{t}, \mathsf{b}, \mathsf{f}\}, \{\mathsf{t}, \mathsf{n}, \mathsf{f}\}, \{\mathsf{b}, \mathsf{n}, \mathsf{f}\}, \{\mathsf{t}, \mathsf{b}, \mathsf{n}, \mathsf{f}\}\Big\}$$

D_0 and D_1 are the sets of designated values of V_0 and V_1, respectively, that is, the values in terms of which the notion of *counterexample* – and, in terms thereof, *consequence relation* – is defined.

5.3. **Plurivalent FDE consequence.** We now define the consequence relation $\vDash_i$ over the set of i valuations (per above, §5) as an absence-of-counterexample relation: $\Sigma \mathrel{.\,\raisebox{1ex}{.}\,.} A$ is i-valid iff there are no i-counterexamples:

$$\Sigma \vDash_i A \text{ iff for all } v_i\text{, either there's some } B \in \Sigma \text{ such that } v_i(B) \notin D_i \text{ or } v_i(A) \in D_i$$

5.4. **On the 'conservativity' of $\vDash_i$ for finite i.** Following Priest's construction [11, p.7] we define the function $\theta : V_1 \to V_0$ by

$$\theta(X) = \begin{cases} \mathsf{b} & \textit{if } \mathsf{b} \in X \textit{ or } \{\mathsf{t}, \mathsf{f}\} \subseteq X \\ \mathsf{t} & \textit{if } \mathsf{t} \in X \textit{ and } \{\mathsf{b}, \mathsf{f}\} \not\subseteq X \\ \mathsf{f} & \textit{if } \mathsf{f} \in X \textit{ and } \{\mathsf{b}, \mathsf{t}\} \not\subseteq X \\ \mathsf{n} & \textit{otherwise.} \end{cases}$$

The map $\theta : V_1 \to V_0$ is surjective. Furthermore, it is easy to show (as Priest proves) that this is a homomorphism. Then by Theorem 1, we have a sort of 'conservativity' result:

Theorem 2. *If $\vDash_0$ is the consequence relation for FDE, and $\vDash_1$ its corresponding plurivalent extension, then*

$$\Sigma \vDash_0 A \textit{ iff } \Sigma \vDash_1 A$$

Priest [11] calls the left-right direction – namely, univalent consequence to plurivalent consequence – of Theorem 2 'conservativity' to reflect that the plurivalent relation keeps the original relation intact.

6. Conservativity of plurivalent FDE in general?

Notice that since any univalent semantics may be naturally embedded into its corresponding plurivalent extension by $\sigma : V_u \to V_p$ defined as $\sigma(x) = \{x\}$, the following always holds:

$$\Sigma \vDash_p A \implies \Sigma \vDash_u A$$

Accordingly, the primary question at hand – driven by the motivating philosophical objection in §1 – is whether for *every* ordinal α the α-plurivalent extension of FDE is conservative with respect to FDE. Priest's work [11, fn 11-12] suggests that the answer is affirmative. After showing definitively that n-plurivalent FDE remains conservative over FDE for any natural number n, Priest states that

[7] It is precisely because some of the other consequence relations discussed in the Appendix to this paper go beyond our target FDE (by deviating sharply from FDE's truth/falsity conditions for connectives) and, in turn, go beyond the driving philosophical argument(s). (The appendix is given only for mathematical completeness.)

> For the limit: anything valid in the limit logic is valid in each finite 'approximation'. Conversely, anything invalid in it is invalid in some approximation, since only finitely many values are employed in the counter-model. [11]

This argument certainly holds for an ω-plurivalent FDE; however, to blunt the target objection against the combinatorial argument for FDE (see §1) we seek to make explicit the target conservativity result for *any* ordinal-plurivalent extension of FDE.

7. Ordinal-Plurivalent Extensions of FDE

We extend the previously stated result to any given ordinal by transfinite induction. Define $\langle V_0, D_0 \neg_0, \wedge_0, \vee_0 \rangle$ as the standard FDE semantics. Then define recursively for all successor ordinals $\alpha + 1$,

$$\begin{aligned} &V_{\alpha+1} = \wp(V_\alpha) \setminus \{\emptyset\} \\ &D_{\alpha+1} = \{X \mid X \cap D_\alpha \neq \emptyset\} \\ &\neg_{\alpha+1} X = \{\neg_\alpha x \mid x \in X\} \\ &X \wedge_{\alpha+1} Y = \{x \wedge_\alpha y \mid x \in X \text{ and } y \in Y\} \\ &X \vee_{\alpha+1} Y = \{x \vee_\alpha y \mid x \in X \text{ and } y \in Y\} \end{aligned}$$

For ease of notation, for each $\beta < \alpha$, identify V_β with $\sigma_{\beta \mapsto \alpha}(V_\beta)$ and D_β with $\sigma_{\beta \mapsto \alpha}(D_\beta)$, where $\sigma_{\beta \mapsto \alpha}$ is the natural embedding from $\langle V_\beta, D_\beta \rangle$ into $\langle V_\alpha, D_\alpha \rangle$. Further we can identify $\wedge_\beta$ as $\wedge_\alpha \restriction \sigma_{\beta \mapsto \alpha}(V_\beta)$, $\neg_\beta$ as $\neg_\alpha \restriction \sigma_{\beta \mapsto \alpha}(V_\beta)$, and $\vee_\beta$ as $\vee_\alpha \restriction \sigma_{\beta \mapsto \alpha}(V_\beta)$. From here we may define for any limit ordinal α,

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta \qquad D_\alpha = \bigcup_{\beta < \alpha} D_\beta \qquad \neg_\alpha = \bigcup_{\beta < \alpha} \neg_\beta$$

$$\wedge_\alpha = \bigcup_{\beta < \alpha} \wedge_\beta \qquad \vee_\alpha = \bigcup_{\beta < \alpha} \vee_\beta$$

7.1. **α-plurivalent FDE consequence.** Consequence in α-plurivalent FDE is defined again in the natural sense: if v is a valuation whose range is V_α then

$$\Sigma \vDash_\alpha A \text{ iff for all } v, \text{ either there's some } B \in \Sigma \text{ such that } v(B) \notin D_\alpha \text{ or } v(A) \in D_\alpha$$

7.2. **Primary result.** Our primary theorem may now be put as follows.

Theorem 3. *If $\vDash_0$ is the consequence relation for FDE, and $\vDash_\alpha$ its corresponding α-plurivalent extension for any ordinal α, then*

$$\Sigma \vDash_\alpha A \textit{ iff } \Sigma \vDash_0 A$$

Proof. We proceed by transfinite induction. Suppose $\alpha + 1$ is a successor ordinal, and that there exists an onto homomorphism from V_β to V_0 for all $\beta < \alpha + 1$. Let θ be the onto homomorphism from $V_\alpha \to V_0$ which exists by this assumption. Extend θ to $V_{\alpha+1} \to V_0$ s.t. for $X \in V_{\alpha+1}$ as:

$$\theta(X) = \begin{cases} \mathsf{b} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = \mathsf{b} \textit{ or } \exists x, y \in X \text{ s.t. } \theta(x) = \mathsf{t} \textit{ and } \theta(y) = \mathsf{f} \\ \mathsf{t} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = \mathsf{t} \textit{ and } \forall y \in X \ \theta(y) \notin \{\mathsf{b}, \mathsf{f}\} \\ \mathsf{f} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = \mathsf{f} \textit{ and } \forall y \in X \ \theta(y) \notin \{\mathsf{b}, \mathsf{t}\} \\ \mathsf{n} & \textit{otherwise.} \end{cases}$$

Lemma 1. *θ is a homomorphism from $V_{\alpha+1}$ onto V_0.*

Proof of Lemma 1. It is obvious that θ is onto. To show that it is a homomorphism it suffices to show that:

i) $\neg_0 \theta(X) = \theta(\neg_{\alpha+1} X)$
ii) $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_{\alpha+1} Y)$
iii) $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_{\alpha+1} Y)$

We prove the lemma by transfinite induction on cases; note that i), ii), and iii) are all obvious if $X \in V_0$.

i) $\neg_0 \theta(X) = \theta(\neg_{\alpha+1} X)$

Suppose $\neg_0 \theta(X) = \theta(\neg_\alpha X)$ for all X, and recall $\theta(\neg_{\alpha+1} X) = \theta(\{z | \exists x_1 \in X \text{ s.t. } z = \neg_\alpha x_1\})$.

(1) Suppose $\theta(X) = \mathsf{b}$ in the case that $\exists x \in X$ s.t. $\theta(x) = \mathsf{b}$. Then:

$$
\begin{aligned}
&\exists z \in \neg_{\alpha+1} X \text{ s.t. } z = \neg_\alpha x && \text{(Defn. of } \neg_{\alpha+1}) \\
&\implies \theta(z) = \theta(\neg_\alpha x) = \neg_0 \theta(x) = \neg_0 \mathsf{b} = \mathsf{b} && \text{(Inductive assumption)} \\
&\implies \theta(\neg_{\alpha+1} X) = \mathsf{b} = \neg_0 \mathsf{b} = \neg_0 \theta(X) && \text{(Defn. of } \theta)
\end{aligned}
$$

Remark. Cases 2 through 4 have been left to the reader. All cases are similar in method.

(5) Suppose $\theta(X) = \mathsf{n}$, i.e. $\forall z' \in X\ \theta(z') = \mathsf{n}$. Then:

$$
\begin{aligned}
&\forall z \in \neg_{\alpha+1} X, z = \neg_\alpha z' \text{ for some } z' \in X && \text{(Defn. of } \neg_{\alpha+1}) \\
&\implies \theta(z) = \theta(\neg_\alpha z') = \neg_0 \theta(z') = \neg_0 \mathsf{n} = \mathsf{n} && \text{(Inductive assumption)} \\
&\therefore \forall z \in \neg_{\alpha+1} X,\ \theta(z) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}\} && \\
&\implies \theta(\neg_{\alpha+1} X) = \mathsf{n} = \neg_0 \mathsf{n} = \neg_0 \theta(X) && \text{(Defn. of } \theta)
\end{aligned}
$$

Therefore, we have that in all cases $\neg_0 \theta(X) = \theta(\neg_{\alpha+1} X)$, so i) is proven. ■ i)

ii) $\theta(X) \wedge_0 \theta(Y) = \theta(x \wedge_{\alpha+1} Y)$

Suppose that for all X, Y, $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_\alpha Y)$ and recall $\theta(X \wedge_{\alpha+1} Y) = \theta(\{z | \exists x_1 \in X, y_1 \in Y \text{ s.t. } z = x_1 \wedge_\alpha y_1\})$. Note that since $\wedge_{\alpha+1}$ is commutative, we need only check 11 cases.

(1) Suppose $\theta(X) = \theta(Y) = \mathsf{b}$ in the case that $\exists x \in X$ s.t. $\theta(x) = \mathsf{b}$ and $\exists y \in Y$ s.t. $\theta(y) = \mathsf{b}$. Then:

$$
\begin{aligned}
&\exists z \in X \wedge_{\alpha+1} Y \text{ s.t. } z = x \wedge_\alpha y && \text{(Defn. of } \wedge_{\alpha+1}) \\
&\implies \theta(z) = \theta(x \wedge_\alpha y) = \theta(x) \wedge_0 \theta(y) = \mathsf{b} \wedge_0 \mathsf{b} = \mathsf{b} && \text{(Inductive assumption)} \\
&\implies \theta(X \wedge_{\alpha+1} Y) = \mathsf{b} = \mathsf{b} \wedge_0 \mathsf{b} = \theta(X) \wedge_0 \theta(Y) && \text{(Defn. of } \theta)
\end{aligned}
$$

(2) Suppose $\theta(X) = \theta(Y) = \mathsf{b}$ in the case that $\exists x \in X$ s.t. $\theta(x) = \mathsf{b}$ and $\exists y_0, y_1 \in Y$ s.t. $\theta(y_0) = \mathsf{t}$ and $\theta(y_1) = \mathsf{f}$. Then:

$$
\begin{aligned}
&\exists z \in X \wedge_{\alpha+1} Y \text{ s.t. } z = x \wedge_\alpha y_0 && \text{(Defn. of } \wedge_{\alpha+1}) \\
&\implies \theta(z) = \theta(x \wedge_\alpha y_0) = \theta(x) \wedge_0 \theta(y_0) = \mathsf{b} \wedge_0 \mathsf{t} = \mathsf{b} && \text{(Inductive assumption)} \\
&\implies \theta(X \wedge_{\alpha+1} Y) = \mathsf{b} = \mathsf{b} \wedge_0 \mathsf{b} = \theta(X) \wedge_0 \theta(Y) && \text{(Defn. of } \theta)
\end{aligned}
$$

Remark. Cases 3 through 10 have been left to the reader. All cases are similar in method.

(11) Suppose $\theta(X) = \mathsf{f}$ or $\theta(Y) = \mathsf{f}$. Without loss of generality, we may assume $\theta(X) = \mathsf{f}$ and $\theta(Y) = \mathsf{t}$. Then $\exists x \in X$ s.t. $\theta(x) = \mathsf{f}$ and $\forall x' \in X\ \theta(x') \notin \{\mathsf{b}, \mathsf{t}\}$. Choose any arbitrary $y \in Y$, and without

loss of generality we may assume $\theta(y) = \mathsf{t}$. Then:

$\exists z \in X \wedge_{\alpha+1} Y$ s.t. $z = x \wedge_\alpha y$ (Defn. of $\wedge_{\alpha+1}$)

$\implies \theta(z) = \theta(x \wedge_\alpha y) = \theta(x) \wedge_0 \theta(y) = \mathsf{f} \wedge_0 \mathsf{t} = \mathsf{f}$ (Inductive assumption)

$\forall z' \in X \wedge_{\alpha+1} Y, \theta(z') \notin \{\mathsf{b}, \mathsf{t}\}$ (Defn. of $\wedge_{\alpha+1}$,$\forall x' \in X, \theta(x') \notin \{\mathsf{b}, \mathsf{t}\}$)

$\implies \theta(X \wedge_{\alpha+1} Y) = \mathsf{f} = \mathsf{f} \wedge_0 \mathsf{t} = \theta(X) \wedge_0 \theta(Y)$ (Defn. of θ)

Therefore, we have that in all cases $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_{\alpha+1} Y)$, so ii) is proven. ■ ii)

iii) $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_{\alpha+1} Y)$

Suppose that for all X, Y, $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_\alpha Y)$ and recall that

$$\theta(X \vee_{\alpha+1} Y) = \theta(X \vee_{\alpha+1} Y) = \theta(\{z | \exists x_1 \in X, y_1 \in Y \text{ s.t. } z = x_1 \vee_\alpha y_1\})$$

Again since $\vee_{\alpha+1}$ is commutative we need only check 11 cases.

(1) Suppose $\theta(X) = \theta(Y) = n$, i.e. $\forall x \in X, y \in Y$ $\theta(x) = \theta(y) = n$. Then:

$\forall z \in X \vee_{\alpha+1} Y, z = x \vee_\alpha y$ for some $x \in X, y \in Y$ (Defn. of $\vee_{\alpha+1}$)

$\implies \theta(z) = \theta(x \vee_\alpha y) = \theta(x) \vee_0 \theta(y) = \mathsf{n} \vee_0 \mathsf{n} = \mathsf{n}$ (Inductive assumption)

$\therefore \forall z \in X \vee_{\alpha+1} Y, \theta(z) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}\}$

$\implies \theta(X \vee_{\alpha+1} Y) = \mathsf{n} = \mathsf{n} \vee_0 \mathsf{n} = \theta(X) \vee_0 \theta(Y)$ (Defn. of θ)

(2) Suppose $\theta(X) = \mathsf{n}$ and $\theta(Y) = \mathsf{f}$, i.e. $\forall x \in X$ $\theta(x) = \mathsf{n}$, $\exists y' \in Y$ s.t. $\theta(y') = \mathsf{f}$ and $\forall y \in Y$ $\theta(y) \notin \{\mathsf{b}, \mathsf{t}\}$.

$\forall z \in X \vee_{\alpha+1} Y, z = x \vee_\alpha y$ for some $x \in X, y \in Y$ (Defn. of $\vee_{\alpha+1}$)

$\implies \theta(z) = \theta(x \vee_\alpha y) = \theta(x) \vee_0 \theta(y) = \mathsf{n} \vee_0 \theta(y) = \mathsf{n}$ (Ind. as., $\forall y \in Y$ $\theta(y) \notin \{\mathsf{b}, \mathsf{t}\}$)

$\therefore \forall z \in X \vee_{\alpha+1} Y, \theta(z) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}\}$

$\implies \theta(X \vee_{\alpha+1} Y) = \mathsf{n} = \mathsf{n} \vee_0 \mathsf{n} = \theta(X) \vee_0 \theta(Y)$ (Defn. of θ)

Remark. Cases 3 through 10 have been left to the reader. As before, all cases are similar in method.

(11) Suppose $\theta(X) = \mathsf{t}$ or $\theta(Y) = \mathsf{t}$. Without loss of generality, we assume that $\theta(X) = \mathsf{t}$ and $\theta(Y) = \mathsf{f}$, so $\exists x' \in X$ s.t. $\theta(x') = \mathsf{t}$ and $\forall x \in X$ $\theta(x) \notin \{\mathsf{b}, \mathsf{f}\}$.

$\exists z' \in X \vee_{\alpha+1} Y$ s.t. $z' = x' \vee_\alpha y$ (Defn. of $\vee_{\alpha+1}$)

$\implies \theta(z') = \theta(x' \vee_\alpha y) = \theta(x') \vee_0 \theta(y) = \mathsf{t} \vee_0 \theta(y) = \mathsf{t}$ (Inductive assumption)

$\forall z \in X \vee_{\alpha+1} Y, \theta(z) \notin \{\mathsf{b}, \mathsf{f}\}$ ($\forall x \in X, \theta(x) \notin \{\mathsf{b}, \mathsf{f}\}$)

$\implies \theta(X \vee_{\alpha+1} Y) = \mathsf{n} = \mathsf{n} \vee_0 n = \theta(X) \vee_0 \theta(Y)$ (Defn. of θ)

Therefore, we have that in all cases $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_{\alpha+1} Y)$, so iii) is proven. ■ iii)

This concludes the proof of Lemma 1. ■

From Lemma 1 we may conclude from Theorem 1 that if $\alpha + 1$ is a successor ordinal then

$$\Sigma \vDash_{\alpha+1} A \text{ iff } \Sigma \vDash_0 A$$

Suppose, now, that α is a *limit* ordinal, and suppose there exists an onto homomorphism from V_β to V_0 for all $\beta < \alpha$. Let γ be the greatest limit ordinal s.t. $\gamma < \alpha$ (or $\gamma = 0$ if no such limit ordinal exists). Now, since for each $\beta < \gamma$, $\langle V_\beta, D_\beta, \neg_\beta, \wedge_\beta, \vee_\beta \rangle$ is embedded in $\langle V_\gamma, D_\gamma, \neg_\gamma, \wedge_\gamma, \vee_\gamma \rangle$ we can redefine the semantics for α values thus:

$$V_\alpha = \bigcup_{\gamma \leq \beta < \alpha} V_\beta \quad D_\alpha = \bigcup_{\gamma \leq \beta < \alpha} D_\beta \quad \neg_\alpha = \bigcup_{\gamma \leq \beta < \alpha} \neg_\beta$$

$$\wedge_\alpha = \bigcup_{\gamma \leq \beta < \alpha} \wedge_\beta \quad \vee_\alpha = \bigcup_{\gamma \leq \beta < \alpha} \vee_\beta$$

By the inductive assumption, θ is already defined and θ is a homomorphism from V_γ onto V_0. Thus to extend θ to all of V_α, define $\theta : V_\beta \to V_0$ as above, recursively for each successor ordinal $\gamma < \beta < \alpha$.

Lemma 2. *θ is a homomorphism from V_α onto V_0.*

Proof of Lemma 2. Fix $X, Y \in V_\alpha$. Then $X, Y \in V_\beta$ for some successor ordinal $\beta < \alpha$. By Lemma 1 we have that

i) $\neg_0 \theta(X) = \theta(\neg_\beta X)$
ii) $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_\beta Y)$
iii) $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_\beta Y)$

Since $X, Y \in V_\alpha$ were arbitrary, and since each of $\neg_\beta, \wedge_\beta$, and $\vee_\beta$ are embedded into $\neg_\alpha, \wedge_\alpha$, and $\vee_\alpha$ for $\beta < \alpha$, we have that for all $X, Y \in V_\alpha$

i) $\neg_0 \theta(X) = \theta(\neg_\alpha X)$
ii) $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_\alpha Y)$
iii) $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_\alpha Y)$

which concludes the proof of Lemma 2. (Our proof here is simply an elucidation for any limit ordinal of the previously noted 'limit' case from Priest [11, fn.12].) ■

From Lemma 2 we may conclude that if α is any limit ordinal

$$\Sigma \vDash_\alpha A \text{ iff } \Sigma \vDash_0 A.$$

Combining this with the result from Lemma 1 completes the proof of Theorem 3. ■

8. Conclusion

Our principal result, which generalizes and makes explicit Priest's previous results, is of direct philosophical value in a defense of the simple but very natural 'combinatorial' argument (see §1) from the standard account of logical consequence to the FDE account. Our aim in this note has not been to argue for or otherwise endorse the given combinatorial argument, but rather only to defend it against the given objection by way of establishing the target result.[8]

[8]Acknowledgements: we're grateful to Joseph Lurie and Benjamin Middleton for comments on an early draft, and to Shay Logan for very helpful comments on a late draft. We also thank Tim McNicholl for providing a remark that a complete combinatorial defense of the objection would require extension to any ordinal.

9. Appendix: Extension to the FDE family of logics

In Priest's previous work [11] he discusses a family of consequence relations beyond FDE. That family, as indicated (in a footnote) on page 4, goes well beyond the philosophical conception of logical consequence briefly sketched in §1, and so we have focused only on our chief target FDE (and its α-plurivalent extensions for any ordinal α). Still, for – but only for – mathematical completeness (not for the driving philosophical objection or defense), we provide this appendix that makes explicit the more general mathematical result (for any ordinal) for what Priest calls 'the FDE family'.[9]

In 'Plurivalent Logics' Priest uses the generalized logic of FDE_φ, of which he notes that all of the following are sublogics of:

- $\emptyset$: classical logic, CL
- e: Bochvar logic (weak Kleene 3-valued logic), B_3
- n: strong Kleene 3-valued logic, K_3
- b: logic of paradox, LP
- en: not yet formulated
- eb: logic of Oller, AL
- bn: first degree entailment, FDE
- bne: FDE_φ

Concerning FDE_φ, we must redefine our base case. Let $V_0 = \{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}, \mathsf{e}\}$, the set of FDE_φ truth values. Here we recall the traditional five-valued semantics of FDE_φ, given as

x	$\neg_0 x$
t	f
b	b
n	n
f	t
e	e

$x \wedge_0 y$	t	b	n	f	e
t	t	b	n	f	e
b	b	b	f	f	e
n	n	f	n	f	e
f	f	f	f	f	e
e	e	e	e	e	e

$x \vee_0 y$	t	b	n	f	e
t	t	t	t	t	e
b	t	b	t	b	e
n	t	t	n	n	e
f	t	b	n	f	e
e	e	e	e	e	e

Again we have $D_0 = \{\mathsf{t}, \mathsf{b}\}$ as the set of designated truth values and given any valuation v whose range is V_0, we define

$$\Sigma \vDash_0^\varphi A \text{ iff for all } v, \exists B \in \Sigma \text{ s.t. } v(B) \notin D_0 \text{ or } v(A) \in D_0$$

Define $\langle V_0, D_0 \neg_0, \wedge_0, \vee_0 \rangle$ as the standard FDE_φ semantics. Then define recursively for each successor ordinal $\alpha + 1$

$$\begin{aligned}
&V_{\alpha+1} = \wp(V_\alpha) \\
&D_{\alpha+1} = \{X \,|\, X \cap D_\alpha \neq \emptyset\} \\
&\neg_{\alpha+1} X = \{\neg_\alpha x \,|\, x \in X\} \\
&X \wedge_{\alpha+1} Y = \{x \wedge_\alpha y \,|\, x \in X, y \in Y\} \\
&X \vee_{\alpha+1} Y = \{x \vee_\alpha y \,|\, x \in X, y \in Y\}
\end{aligned}$$

Again recall $\sigma(x) = \{x\}$ for any x. And again for ease of notation we can now, for each ordinal $\beta < \alpha$, identify V_β with $\sigma_{\beta \mapsto \alpha}(V_\beta)$ and D_β with $\sigma_{\beta \mapsto \alpha}(D_\beta)$. Further we can identify $\wedge_\beta$ as $\wedge_\alpha \restriction \sigma_{\beta \mapsto \alpha}(V_\beta)$, $\neg_\beta$ as $\neg_\alpha \restriction \sigma_{\beta \mapsto \alpha}(V_\beta)$, and $\vee_\beta$ as $\vee_\alpha \restriction \sigma_{\beta \mapsto \alpha}(V_\beta)$. We can now define α-plurivalent FDE_φ for any limit ordinal α:

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta \quad D_\omega = \bigcup_{\beta < \alpha} D_\beta \quad \neg_\omega = \bigcup_{\beta < \alpha} \neg_\beta$$

$$\wedge_\omega = \bigcup_{\beta < \alpha} \wedge_\beta \quad \vee_\omega = \bigcup_{\beta < \alpha} \vee_\beta$$

Consequence is defined again in the natural sense: if v is a valuation whose range is V_ω then

$$\Sigma \vDash_\omega^\varphi A \text{ iff for all } v, \exists B \in \Sigma \text{ s.t. } v(B) \notin D_\omega \text{ or } v(A) \in D_\omega$$

[9]While Priest's name 'FDE family' is motivated from a mathematical perspective, it is arguably unfortunate given that some in the family go directly against the FDE truth/falsity conditions for connectives – something that, as we said in §5, is critical to FDE (and its proper extensions). But we flag this terminological issue only to drop it.

Theorem 4. *If $\vDash_0^\varphi$ is the consequence relation for FDE_φ, and $\vDash_\alpha^\varphi$ its corresponding α-plurivalent extension for any ordinal α, then*

$$\Sigma \vDash_\alpha^\varphi A \textit{ iff } \Sigma \vDash_0^\varphi A$$

Proof. Fix an ordinal α. We extend $\theta : V_\alpha \to V_0$ to FDE_φ as follows. Define recursively for successor ordinals $\beta < \alpha$, and $X \in V_\beta$:

$$\theta(X) = \begin{cases} \mathsf{b} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = \mathsf{b} \textit{ or } \exists x, y \in X \text{ s.t. } \theta(x) = \mathsf{t} \textit{ and } \theta(y) = \mathsf{f} \\ \mathsf{t} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = \mathsf{t} \textit{ and } \forall y \in X\ \theta(y) \notin \{\mathsf{b}, \mathsf{f}\} \\ \mathsf{f} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = \mathsf{f} \textit{ and } \forall y \in X\ \theta(y) \notin \{\mathsf{b}, \mathsf{t}\} \\ \mathsf{n} & \textit{if } \exists x \in X \textit{ s.t. } \theta(x) = n \textit{ and } \forall y \in X\ \theta(y) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}\} \\ \mathsf{e} & \textit{otherwise.} \end{cases}$$

We now prove the following:

Lemma 3. *θ is a homomorphism from V_α onto V_0.*

Proof: It is again obvious that θ is onto. By the same reasoning as in the main argument, it suffices show that θ is a homomorphism for any successor ordinal $\alpha + 1$, i.e.

i) $\neg_0 \theta(X) = \theta(\neg_{\alpha+1} X)$
ii) $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_{\alpha+1} Y)$
iii) $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_{\alpha+1} Y)$

We now prove the lemma by transfinite induction on cases; note that i), ii), and iii) are all obvious if $X \in V_0$. Notice further that all the cases from Lemma 1 still hold true here, which is obvious from FDE being a sublogic of FDE_φ. We therefore only consider the cases where at least one of $\theta(X) = e$ or $\theta(Y) = e$.

i) $\neg_0 \theta(X) = \theta(\neg_{\alpha+1} X)$

Suppose $\neg_0 \theta(X) = \theta(\neg_\alpha X)$ for all X, and recall $\theta(\neg X) = \theta(\neg_{\alpha+1} X) = \theta(\{z | \exists x \in X \text{ s.t. } z = \neg_\alpha x\})$.

(6) Suppose $\theta(X) = \mathsf{e}$, i.e. $\forall x \in X$, $\theta(x) = \mathsf{e}$. Then:

$$\begin{array}{ll} \forall z \in \neg_{\alpha+1} X, z = \neg_\alpha x \text{ for some } x \in X & \text{(Defn. of } \neg_{\alpha+1}) \\ \implies \theta(z) = \theta(\neg_\alpha x) = \neg_0 \theta(x) = \neg_0 \mathsf{e} = \mathsf{e} & \text{(Inductive assumption)} \\ \therefore \forall z \in \neg_{\alpha+1} X, \theta(z) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}, \mathsf{n}\} & \\ \implies \theta(\neg_{\alpha+1} X) = \mathsf{e} = \neg_0 \mathsf{e} = \neg_0 \theta(X) & \text{(Defn. of } \theta) \end{array}$$

Therefore, we have that in all cases $\neg_0 \theta(X) = \theta(\neg_{\alpha+1} X)$, so i) is proven. ■ i)

ii) $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_{\alpha+1} Y)$

Suppose that for all X, Y, $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_\alpha Y)$ and recall $\theta(X \wedge_{\alpha+1} Y) = \theta(\{z | \exists x \in X, y \in Y \text{ s.t. } z = x \wedge_\alpha y\})$.

(12) Without loss of generality, suppose $\theta(Y) = \mathsf{e}$, i.e. $\forall y \in Y$, $\theta(y) = \mathsf{e}$. Then:

$$\begin{array}{ll} \forall z \in X \wedge_{\alpha+1} Y, z = x \wedge_\alpha y \text{ for some } x \in X, y \in Y & \text{(Defn. of } \wedge_{\alpha+1}) \\ \implies \theta(z) = \theta(x \wedge_\alpha y) = \theta(x) \wedge_0 \theta(y) = \theta(x) \wedge_0 \mathsf{e} = \mathsf{e} & \text{(Inductive assumption)} \\ \therefore \forall z \in X \wedge_{\alpha+1} Y, \theta(z) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}, \mathsf{n}\} & \\ \implies \theta(X \wedge_{\alpha+1} Y) = \mathsf{e} = \theta(X) \wedge_0 \mathsf{e} = \theta(X) \wedge_0 \theta(Y) & \text{(Defn. of } \theta) \end{array}$$

Therefore, we have that in all cases $\theta(X) \wedge_0 \theta(Y) = \theta(X \wedge_{\alpha+1} Y)$, so ii) is proven. ■ ii)

iii) $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_{\alpha+1} Y)$

Suppose that for all X, Y, $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_\alpha Y)$ and recall $\theta(X \vee Y) = \theta(X \vee_{\alpha+1} Y) = \theta(\{z | \exists x \in X, y \in Y \text{ s.t. } z = x \vee_\alpha y\})$.

(12) Without loss of generality, suppose $\theta(Y) = \mathsf{e}$, i.e. $\forall y \in Y$, $\theta(y) = \mathsf{e}$. Then:

$$\begin{aligned}
&\forall z \in X \vee_{\alpha+1} Y, z = x \vee_\alpha y \text{ for some } x \in X, y \in Y && \text{(Defn. of } \vee_{\alpha+1}) \\
&\implies \theta(z) = \theta(x \vee_\alpha y) = \theta(x) \vee_0 \theta(y) = \theta(x) \vee_0 \mathsf{e} = \mathsf{e} && \text{(Inductive assumption)} \\
&\therefore \forall z \in X \vee_{\alpha+1} Y, \theta(z) \notin \{\mathsf{b}, \mathsf{t}, \mathsf{f}, \mathsf{n}\} \\
&\implies \theta(X \vee_{\alpha+1} Y) = \mathsf{e} = \theta(X) \vee_0 \mathsf{e} = \theta(X) \vee_0 \theta(Y) && \text{(Defn. of } \theta)
\end{aligned}$$

Therefore, we have that in all cases $\theta(X) \vee_0 \theta(Y) = \theta(X \vee_{\alpha+1} Y)$, so iii) is proven. ■ iii)

This concludes the proof of the Lemma 3, and therefore the proof of Theorem 4. ■

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