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Multiple-conclusion LP and default classicality
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Abstract. Philosophical applications of familiar paracomplete and paraconsistent logics often rely on an idea of 'default classicality'. With respect to the paraconsistent logic LP (ithe dual of Strong Kleene or K3), such 'default classicality' is standardly cashed out via an LP-based non-monotonic logic due to Priest (1991, 2006a). In this paper, I offer an alternative approach via a monotonic, multiple-conclusion version of LP.

§1. Introduction. The logic LP is the dual of the well-known logic K3 (viz., Strong Kleene).[1] This logic, like K3, has found prominent applications in philosophy, particularly with respect to paradoxical phenomena (Beall, 2009; Brady, 2006; Field, 2008; Kripke, 1975; Priest, 2006a,b). In such applications, the background picture is one of 'default classicality'. The basic thought is that classical logic is 'right' (in some sense) for the broad array of 'normal' cases; however, various 'abnormal' (e.g., paradoxical) phenomena motivate a slightly weaker logic. In short, the thought is that *classical logic is the default logic, and the weaker logic kicks into gear when necessary.*

The chief question is how to understand this 'default classicality', particularly in the LP case. We may distinguish two (closely related) questions:

- How, if at all, does the *logic* reflect default classicality?
- How does *use* of the logic reflect *defeasibility*?

In the K3 case, the answers are relatively clear: adding appropriate premises of the form $A \vee \neg A$ in effect collapses K3 into classical logic; and one's *use* of the logic invokes extra-logical principles that generally warrant the additional premises. But LP is different, at least with respect to the first question: there is no corresponding *adding-to-the-premises* recipe for LP that yields the target 'collapse'. (This will be apparent from the account of LP in §2.1.) And so another route towards the questions is required for LP theorists.

The standard route, advanced by Priest (1991, 2006b), answers both questions by constructing an LP-based *non-monotonic logic*, namely, *minimally inconsistent LP*. This logic both formally models default classicality and similarly purports to thereby model 'defeasible reasoning'.

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[1] LP was first discussed, under a different label, by Asenjo (1966), but later independently discovered and widely advanced by Priest (1979, 1984). A more leisurely discussion of K3, LP, and a familiar family of non-classical logics is widely available (Beall, 2010; Beall & van Fraassen, 2003; Priest, 2008; Restall, 2005a). While much of what I say applies, dually, to the K3 case—e.g., a dual version of Theorem 3.13 is available—my own paraconsistent interests (Beall, 2009) guide my focus here on LP.

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I have no substantive objections to the non-monotonic approach, but I nonetheless propose a different—and perhaps simpler—approach in terms of a *monotonic*, *multiple-conclusion* logic that I shall call LP^+.[2] The thought is simple: just as the K3 theorist adds appropriate *premises* of the form $A \vee \neg A$ to 'return to classical logic', the LP^+ theorist does the dual—namely, adds appropriate sentences of the form $A \wedge \neg A$ to the *conclusion* set.

In §2. I present LP^+. Towards answering the first question concerning default classicality, §3. presents a simple though crucial theorem concerning a relation between LP^+ and CPL^+ (i.e., the corresponding multiple-conclusion version of classical propositional logic). §4. takes up the second question concerning default consistency and *reasoning*, suggesting a general sense in which multiple-conclusion logic(s) leave us with 'defeasible *inferences*'.

§2. Multiple-conclusion LP. I first present (single-conclusion) LP, and then turn to the multiple-conclusion generalization LP^+. Because of the simplicity of the LP model theory, I rely throughout on 'semantic' characterizations of the logics (except in the appendix, where I briefly present an adequate sequent system).

The syntax (throughout) is that of standard CPL (i.e., classical propositional logic), where atomics are propositional variables.[3] Throughout, I let S be the set of all sentences, p any atomic in S, and A any (atomic or molecular) element of S, with $\neg$ (unary), $\vee$ (binary) and, with redundancy, $\wedge$ (binary) the only primitive connectives—with other standard binary connectives (e.g., $\supset$, $\equiv$) defined as usual.

2.1. LP. The (single-conclusion) logic LP may be characterized 'semantically' as follows. Our sentences are interpreted via all (total) valuations $v : S \longrightarrow \{1, .5, 0\}$ that obey the following clauses:

- $v(\neg A) = 1 - v(A)$
- $v(A \vee B) = \max\{v(A), v(B)\}$
- $v(A \wedge B) = \min\{v(A), v(B)\}$

We let $\mathbb{V}_{lp}$ be the set of all such LP valuations.

Definition 2.1 *A valuation $v \in \mathbb{V}_{lp}$* satisfies *A iff $v(A) \in \{1, .5\}$.*

Definition 2.2 *A valuation $v \in \mathbb{V}_{lp}$* satisfies *a set $X \subseteq S$* of sentences *iff v satisfies* every *$A \in X$.*

With these definitions in hand, we define the validity relation—for present purposes, the *logic*—in the usual way.

Definition 2.3 (LP) *$X \vDash_{lp} A$ iff every $v \in \mathbb{V}_{lp}$ that satisfies X satisfies A.*

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[2] On terminology: a *monotonic logic* is one such that adding to the premise set does not take you from a valid argument to an invalid argument. For present purposes, I simply generalize this notion of monotonicity to cover multiple-conclusion: adding to *either* a premise set *or* a conclusion set doesn't take you from a valid argument to an invalid argument. (I do not dwell on this definition, but assume it in the background.)

[3] I ignore (minor) complexities involved in adding predicates or names. While I focus on the propositional level in this paper, the discussion—and results—should carry over to the predicate-cum-quantifier level (though proofs would require more complexity).

2.2. A few notable features. Here, I briefly note some of the distinguishing features of LP. (The reader may consider the dual K3 features for comparison.)

2.2.1. Notable invalidities. LP is a paraconsistent logic: $A \wedge \neg A \nvDash_{lp} B$. Any valuation v such that $v(A) = 0.5$ and $v(B) = 0$ serves as a counterexample.

Moreover, disjunctive syllogism fails in LP: $\neg A, A \vee B \nvDash_{lp} B$. Again, the above counterexample suffices: $v(A) = 0.5$ and $v(B) = 0$.

Similarly, 'material modus ponens' fails, where $A \supset B$ is the material conditional defined as $\neg A \vee B$. In particular, we have $A, A \supset B \nvDash_{lp} B$. The counterexample (mentioned twice) above suffices again here.

2.2.2. Notable validities. While LP is paraconsistent, it is not paracomplete. That LP is not paracomplete comes from the fact that $B \vDash_{lp} A \vee \neg A$.

An important fact, related to the 'default-classicality' idea, is that LP is a sublogic of CPL.

Fact 1 *If* $X \vDash_{lp} A$ *then* $X \vDash_{cpl} A$.

That this is so is evident from the fact that, if you remove the middle value from LP's set of semantic values (i.e., co-domain of the LP valuations), the resulting set of valuations is the familiar set $\mathbb{V}_{cpl}$ of CPL valuations.

2.3. LP$^+$. We achieve LP$^+$, the *multiple-conclusion generalization of LP*, by generalizing the validity relation from the standard *set-to-sentence* relation to a *set-to-set* relation. Instead of validity being a subset of $\wp(S) \times S$, we now take it to be a binary relation on $\wp(S)$, a subset of $\wp(S) \times \wp(S)$.[4]

The semantics for LP is all we need to define LP$^+$, but some additional definitions are convenient:

Definition 2.4 *A valuation* $v \in \mathbb{V}_{lp}$ dissatisfies A *iff* $v(A) = 0$.

Definition 2.5 *A valuation* $v \in \mathbb{V}_{lp}$ dissatisfies a set $X \subseteq S$ of sentences *iff* v *dissatisfies* every $A \in X$.

With these definitions in hand, we define the validity relation—for present purposes, the *logic*—in the usual way, where, as throughout, X and Y are subsets of the set S of sentences.

Definition 2.6 (LP$^+$) $X \vDash^+_{lp} Y$ *iff no* $v \in \mathbb{V}_{lp}$ *satisfies* X *and dissatisfies* Y.

2.3.1. Notable invalidities. All of the single-conclusion invalidities remain, where a single-conclusion argument is one in which the conclusion set is a singleton. Any philosophical virtues of LP, the single-conclusion variant of LP$^+$, thus carry over into *LP*$^+$, which remains paraconsistent and non-paracomplete.[5]

[4] This is simply the standard multiple-conclusion insight, which enjoys often-noted mathematical elegance over its single-conclusion counterpart. My interest here is in some of the philosophical applications of the insight, particularly to the idea of *default classicality* for target non-classical logics.

[5] In this setting, we say that a logic $\vdash$ is *paraconsistent* just if $\Gamma, A, \neg A \nvdash B, \Delta$, and *paracomplete* just if $\Gamma, B \nvdash A, \neg A, \Delta$, where Γ and Δ are subsets of S (sentences).

2.3.2. Notable validities. What is notable—and explicitly noted in Theorem 3.13 below—is that for each of the noted invalidities in LP, there is a corresponding LP$^+$ *validity* that arises from, in effect, treating the 'inconsistency' or 'gluttiness' of one of the premises (i.e., one of the elements of the premise set) as an element of the conclusion set—one of the 'options', so to speak, that the premise set yields in the conclusion set. In particular, we have the following validities corresponding to the LP invalidities noted in §2.2.1. (Proofs are left as exercise.)

- $\{A \wedge \neg A\} \nvDash_{lp} B$ but $\{A \wedge \neg A\} \vDash^+_{lp} \{B, A \wedge \neg A\}$.
- $\{\neg A, A \vee B\} \nvDash_{lp} B$ but $\{\neg A, A \vee B\} \vDash^+_{lp} \{B, A \wedge \neg A\}$.
- $\{A, A \supset B\} \nvDash_{lp} B$ but $\{A, A \supset B\} \vDash^+_{lp} \{B, A \wedge \neg A\}$.

These validities reflect a pattern captured more generally in a simple but, for target philosophical purposes, central theorem concerning CPL$^+$ and LP$^+$.

§3. Default classicality: CPL$^+$ and LP$^+$.

CPL may be defined by taking the 'semantics' for LP and fixing the set $\mathbb{V}_{cpl}$ of CPL valuations to a proper subset of the set $\mathbb{V}_{lp}$ of LP valuations. In particular, $\mathbb{V}_{cpl}$ is simply the set of LP valuations whose range is $\{1, 0\}$. With this in mind, the generalization to CPL$^+$, multiple-conclusion CPL, is straightforward.

3.1. CPL$^+$. The multiple-conclusion generalization of CPL is achieved in the same way that the corresponding generalization to LP$^+$ is achieved. In fact, all of the foregoing definitions apply; and the definition of validity is as follows.

Definition 3.7 (CPL$^+$) *$X \vDash^+_{cpl} Y$ iff no $v \in \mathbb{V}_{cpl}$ satisfies X and dissatisfies Y.*

For present purposes, what is important about this generalized form of classical (propositional) logic is its relation to the corresponding generalization of LP. In particular, that classical logic, so understood (in multiple-conclusion form), is the *default* logic of—the otherwise all-purpose, weaker—logic LP$^+$ is conspicuous from the result below.

3.2. CPL$^+$ and LP$^+$. Throughout, we let X be any subset of S, and p any atomic in S. We define (standard) notions of, respectively, a valuation restricted to a subset of its domain; a set of subsentences of a sentence; a set of subsentences of a set of sentences; the set of atomic sentences in a set of sentences; and an 'atomic inconsistency set' corresponding to a set of sentences:

Definition 3.8 *We let $v{\restriction}_Z$ be the restriction of v to Z as standardly defined.*

Definition 3.9 *Let $\sigma(A)$ be the set of subsentences of A, standardly defined.*

Definition 3.10 *Let $\sigma(X)$ be $\{B : B \in \sigma(A)$ for each $A \in X\}$.*

Definition 3.11 *Let $\alpha(X) = \{p : p$ is atomic and $p \in \sigma(X)\}$.*

Definition 3.12 *Let $\iota(X) = \{p \wedge \neg p : p \in \alpha(X)\}$.*

An example, with respect to Definition 3.12, is $\iota(X) = \{p \wedge \neg p, q \wedge \neg q, r \wedge \neg r\}$, which is the 'atomic inconsistency set' corresponding to any set X for which $\alpha(X) = \{p, q, r\}$, for example $X = \{\neg p, \neg(r \vee q)\}$ or the like.

Theorem 3.13 $X \vDash^+_{cpl} Y$ *iff* $X \vDash^+_{lp} Y \cup \iota(X)$.

Proof.

RLD. This follows immediately from two facts:

Fact 2 $\mathbb{V}_{cpl} \subseteq \mathbb{V}_{lp}$.

Fact 3 *If $\iota(X) \neq \emptyset$, then every $v \in \mathbb{V}_{cpl}$* dis*satisfies $\iota(X)$, that is, no CPL valuation satisfies anything in $\iota(X)$. If $\iota(X) = \emptyset$, then $Y \cup \iota(X) = Y$, and so Fact 2 suffices for the result.*

LRD. Suppose that $X \vDash^{+}_{cpl} Y$ but, for reductio, $X \nvDash^{+}_{lp} Y \cup \iota(X)$, in which case there's some $v \in \mathbb{V}_{lp}$ such that v satisfies X but v *dis*satisfies $Y \cup \iota(X)$. If $v \in \mathbb{V}_{cpl}$, then, by the initial supposition, v satisfies something in Y and, hence, satisfies something in $Y \cup \iota(X)$. So, $v \in \mathbb{V}_{lp} \setminus \mathbb{V}_{cpl}$. Now, either there's some $v' \in \mathbb{V}_{cpl}$ such that $v{\restriction}_{\alpha(X)} = v'{\restriction}_{\alpha(X)}$ or not. In the latter case, $v(p) = 0.5$ for some $p \in \alpha(X)$, and so $v(p \wedge \neg p) = 0.5$ by LP (semantic) conditions; and so v satisfies something in $\iota(X)$ and, in turn, satisfies something in $Y \cup \iota(X)$. Contradiction. In the former case, where there is some $v' \in \mathbb{V}_{cpl}$ such that $v{\restriction}_{\alpha(X)} = v'{\restriction}_{\alpha(X)}$, we immediately get $v{\restriction}_{X} = v'{\restriction}_{X}$ by the following fact:

Fact 4 *Let $v \in \mathbb{V}_{lp}$ and $v' \in \mathbb{V}_{cpl}$. For any $X \subseteq S$, if $v{\restriction}_{\alpha(X)} = v'{\restriction}_{\alpha(X)}$, then $v{\restriction}_{X} = v'{\restriction}_{X}$. (Proof: exercise.)*

But, then, since v satisfies X, so too does v', in which case, since $v' \in \mathbb{V}_{cpl}$ and $X \vDash^{+}_{cpl} Y$, we have that v' satisfies something in Y and, hence, v' satisfies something in $Y \cup \iota(X)$. But, then, so does v. Contradiction. $\square$

Parenthetical remark. I am grateful to Graham Priest who, in comments on an earlier draft, pointed to a related result that he establishes concerning single-conclusion LP (Priest, 2006b, Ch. 8). The proof of Priest's result proceeds via the metatheory (invoking, e.g., compactness and the material-conditional deduction theorem). Given that LP may be seen as the limit (*singleton*-conclusion) case of LP^{+}, the purely semantic proof that I give for Theorem 3.13 extends to LP.

I should also note that Priest's result is related to one discussed by Belnap & Dunn (1973), which turns on introducing a sentential constant f that, informally, may be thought of as the disjunction of all elements of $\iota(S)$. Both results are more coarse-grained than Theorem 3.13; and one advantage of LP^{+} and Theorem 3.13 emerges along these lines by an objection that Belnap and Dunn present against using such (single-conclusion) results for cashing out the 'default classicality' idea:

> the relevantist [or, for present purposes, LP theorist] generally has more information than a barren disjoined f; he knows, if he has done his homework, which contradiction is at issue. ....[A]nd so for him, using f, whether suppressed or not, is to lose information. (Anderson et al., 1992, p. 505)

In the LP^{+} context, we get more fine-grained information via the already-available $\iota(X)$ in question. (And we could define a 'minimally inconsistent conclusion set' to get even finer-grained information.) Of course, one could similarly add a multitude of more finely grained 'atomic-inconsistency constants' (so to speak) to play the role of the already-available family of atomic inconsistency sets $\iota(X)$, but it is not clear what would be achieved over the LP^{+} framework. My own view is that the

LP$^+$ framework is more natural, but 'natural' in a sense that, regrettably, I cannot as yet make precise. I leave the matter for future debate. *End remark.*

3.3. Philosophical application. Theorem 3.13 gives an answer to the first question about default-classicality (without invoking a non-monotonic logic). In short, the theorem above makes plain that CPL$^+$ is our 'default logic' in the sense that, *except when one or more of the premises is a 'glut'*, what follows from the premise set is precisely what follows according to classical logic (here conceived in multiple-conclusion terms).

§4. Choices: default-classical reasoning. The import of Theorem 3.13 is that the consequences of a premise set are the classical consequences—*unless* some of the premises are glutty. This is the sense in which the *logic* exhibits the default-classicality idea.

There is, however, a second question: how is such 'default classicality' or 'default consistency' to be understood with respect to *using the logic*? How does such 'default classicality' show up in *reasoning*? This question, I suggest, has a very natural answer that coincides nicely with a perspective on *using* multiple-conclusion logic in general.

4.1. Logic and choices. Let us assume that logic—or validity—is fundamentally along multiple-conclusion lines, with single(-ton)-conclusion logic the limiting case. What our logic tells us, then, is what *sets* of sentences follow from what sets of sentences.

How, then, shall we understand what logic provides? I suggest that what logic often gives us is a (conclusion) set of *choices*. The logic simply tells you that Y follows from X. What you do with the elements—the options—in Y is beyond logic's rule. (More on this in 4.2.) Of course, given the monotonicity of our logic,[6] talk of 'choices' is interesting chiefly in what might be called *strict-choice validities*, that is, a valid argument $\langle X, Y\rangle$ such that there's no $Z \subset Y$ such that X implies Z. For example, in our target LP$^+$ case,

$$\langle\{\neg q, p \supset q\}, \{\neg p, q \wedge \neg q\}\rangle$$

is a strict-choice validity: it is LP$^+$-valid, but the given premise set fails to imply any proper subset of the conclusion set. It is in strict-choice validities that choices are 'real' (so to speak), though the general suggestion is that all valid arguments provide choices (however degenerate), namely, whatever is in the conclusion set.[7]

4.2. Choices and extra-logical principles. How, then, do we *use* logic on this picture? How do we *draw a conclusion*—make a single *choice*—from a (conclusion) set of choices? The natural answer is a familiar one: we rely on *extra-logical* principles—principles of rationality, pragmatic principles, epistemic princi-

[6] Recall that monotonicity in this context involves both sides—a fortiori, the conclusion-set side. (In proof-theoretic terms, the present point may be made via *weakening on the right*.) I am grateful to an anonymous referee for useful comments on this section.

[7] Along these lines, one might think of logical truths as *dull 'choices'*, namely, those A such that $\{A\} \cup Y$ follows from every (including empty) premise set X, for any $Y \subseteq S$. In LP, such A are precisely the classical logical truths (Priest, 1979); and—as a simple proof shows—the situation is the same for LP$^+$.

ples, or more.[8] And this is precisely where the default-classicality—or defeasible-consistency—comes into play in *using* logic: extra-logical principles.

When logic gives us a strict choice (i.e., delivers a strict-choice validity), a consistency assumption—perhaps based on rationality, perhaps on something else—guides the choice: *reject inconsistency (other things being equal)*. While the details of such principles might be complicated, the general idea is simple. Why, for example, do we normally infer or 'choose' q via the premise set $\{\neg p, p \vee q\}$ if, as I've suggested, our logic delivers only the (conclusion) set $\{q, p \wedge \neg p\}$? The answer—at least in rough form—invokes extra-logical principles: *as a first go, reject the inconsistent options!* Of course, when, in extraordinary (e.g., paradoxical) cases, our choices keep hitting against evidence for inconsistency, we then return to our original conclusion set of choices, and choose a different option. Such is the defeasibility of inference; such is the defeasibility of inquiry. Logic itself can only take us to our options; it leaves extra-logical principles to guide our choices. Logic itself is monotonic; what we do with the choices that logic gives us is defeasible.

§5. Concluding remarks. LP^+, the multiple-conclusion generalization of LP, reflects the idea of default classicality in a natural—and monotonic—way. With respect to the formal logic itself, LP^+ reflects default classicality via Theorem 3.13. In short: except for various 'glutty' or inconsistent phenomena (e.g., paradoxes), what follows from a premise set are precisely its classical consequences.

Logic, qua fundamentally a set-to-set (multiple-conclusion) relation, tells us what *sets* follow from what *sets*. But this often leaves us with choices: a single premise set may leave us with the choice between a consistent and an inconsistent option. And here is where the default-classicality feature of *reasoning*—or *using a logic*—shows up. Logic often leads us to a choice; it's only via *extra-logical principles* that we make our choices. And such principles generally preach against inconsistency. It is only when—in the face of other theoretical pressures (e.g., simplicity, coherence, faithfulness to data, whathaveyou)—inconsistency cannot be avoided that we return to the initial stock of choices and choose an inconsistent option. Such defeasibility is *not* something that the formal logic itself exhibits; such defeasibility is instead a feature of extra-logical principles guiding theoretical inquiry in general.

[8] Lessons of Lewis Carroll (1895) might be construed along these lines, as well as Gilbert Harman's distinction between logic and inference/reasoning (Harman, 1986). Extra-logical principles of rationality—concerning rejection (Field, 2008; Priest, 2006b; Restall, 2005b) or the like—are common in applications of non-classical logics, though formulating such principles can be hard, as Restall (2004), Priest (2006b)[Ch. 19], and particularly Field (2010)—concerning degrees of belief in multiple-conclusion setting—make plain.

Appendix: a sequent system for LP^+. There are various ways to achieve an adequate sequent calculus for LP^+.[9] David Ripley (2011) employs ideas from Baaz et al. (a,b) to construct a 3-sided sequent system for LP that, with minor tweaks, yields a 3-sided set-set system for LP^+.

An alternative approach towards a fully 2-sided system is to take widely-known tagged tableau systems for LP (Beall & Ripley, 2011; Beall & van Fraassen, 2003; Priest, 2008; Restall, 2005a) and use them to construct a corresponding (cut-free) system for LP^+.[10] In particular, use the 'positive' tableau tag (say, '+') for left position and the 'negative' tag (say, '−') for right. For example, the 'positive negated conjunction' rule in such systems directly reflects De Morgan equivalence:

$$\begin{array}{c} \neg(A \wedge B),\ + \\ \neg A \vee \neg B,\ + \end{array}$$

And the same goes for the 'negative negated conjunction' rule:

$$\begin{array}{c} \neg(A \wedge B),\ - \\ \neg A \vee \neg B,\ - \end{array}$$

As above, 'translating' these rules into 2-sided sequent rules puts the 'positive' on the left and the 'negative' on the right (and flipping the top-down tableau order to arrive at our target $\neg\wedge$ operation in the corresponding sequent rules):

$$\neg\wedge \text{ Left: } \frac{\Gamma, \neg A \vee \neg B \ \vdash \ \Delta}{\Gamma, \neg(A \wedge B) \ \vdash \ \Delta} \quad \neg\wedge \text{ Right: } \frac{\Gamma \ \vdash \ \neg A \vee \neg B, \Delta}{\Gamma \ \vdash \ \neg(A \wedge B), \Delta}$$

In turn, the adequacy results of the given tableau system carry over, mutatis mutandis, to the 'generated' sequent system. I briefly sketch such a system here, omitting (routine) adequacy results.

0.1. Notation Throughout, A and B are any sentences unless otherwise noted; Γ, Δ, Π and Σ are any sets (not multisets) of sentences; and, following convention, the comma is union and 'Γ, A' abbreviates '$\Gamma \cup \{A\}$'. I use the turnstile for sequents.

0.2. Axioms

A1. Identity: $\Gamma, A \vdash A, \Delta$, where A is any sentence.[11]
A2. Exhaustion: $\Gamma \vdash A, \neg A, \Delta$, where A is any atomic.[12]

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[9] I am grateful to a referee for suggesting the inclusion of a sequent system in this paper. I am also grateful to Dave Ripley who not only gave me access to his unpublished work on 3-sided systems, but also independently discovered the tableau-to-gentzen system below. Indeed, any novelty in 'my' system below is to be credited equally to Ripley.

[10] I assume familiarity with the target tableau systems here. See any of the cited sources for details, but for the target adequacy results I rely specifically on the system presented by Priest (2008).

[11] An alternative approach is to formulate Identity for *all literals*, and show that it holds for all sentences; however, a direct 'translation' of the target tableau system (Priest, 2008), on whose adequacy results I rely, takes Identity for *all* sentences as primitive. (NB: that one needs to take it as primitive at least for all literals is a feature of the non-classical negation at work.)

[12] A negation $\neg$ connective is sometimes said to be *exhaustive* just when its version of excluded middle holds, that is, just when $A \vee \neg A$ is valid. The role of our exhaustion axiom here is to ensure LP^+'s exhaustive negation.

0.3. Operational Rules What is peculiar about negation in LP$^+$ is its interaction with other connectives. Classical rules are fine for conjunction and disjunction; it's in negation's interaction with such connectives where the non-classicality emerges. All of this is reflected directly in the familiar tableau system(s) for LP (Priest, 2008), which have independent De Morgan rules governing negation's interaction with other boolean connectives. (Sometimes, the De Morgan equivalences are given explicitly in the rules; sometimes they're implied, where a 'positive' rule for negated conjunctions might directly branch into the negated disjuncts, rather than to the corresponding disjunction itself.) The rules below simply rewrite such tableau rules in 2-sided set-set sequent form.

0.3.1. Classical $\wedge$ rules

$$\wedge \text{ Left: } \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad \wedge \text{ Right: } \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

0.3.2. Classical $\vee$ rules

$$\vee \text{ Left: } \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \quad \vee \text{ Right: } \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

0.3.3. Negated conjunctions

$$\neg\wedge \text{ Left: } \frac{\Gamma, \neg A \vee \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta} \quad \neg\wedge \text{ Right: } \frac{\Gamma \vdash \neg A \vee \neg B, \Delta}{\Gamma \vdash \neg(A \wedge B), \Delta}$$

0.3.4. Negated disjunctions

$$\neg\vee \text{ Left: } \frac{\Gamma, \neg A \wedge \neg B \vdash \Delta}{\Gamma, \neg(A \vee B) \vdash \Delta} \quad \neg\vee \text{ Right: } \frac{\Gamma \vdash \neg A \wedge \neg B, \Delta}{\Gamma \vdash \neg(A \vee B), \Delta}$$

0.3.5. Negated negations

$$\neg\neg \text{ Left: } \frac{\Gamma, A \vdash \Delta}{\Gamma, \neg\neg A \vdash \Delta} \quad \neg\neg \text{ Right: } \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg\neg A, \Delta}$$

0.4. Structural rules Since we're using sets, we rely on the (free) rules of contraction and permutation. Cut, which is eliminable, is a rule:

$$\text{Cut: } \frac{\Gamma \vdash \Delta, A \qquad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

Weakening rules, namely,

$$\text{Weakening Left: } \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \text{Weakening Right: } \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$$

are both eliminable for a standard reason: the 'nature' of our sequents—the axioms governing them—already allow side premises (antecedents) and side consequents (succeedents).

0.5. Validity. We say that $\Gamma \vdash \Delta$ is *valid* just if derivable via the above rules.

0.6. Some results.

Theorem 0.14 (Adequacy) *$\Gamma \vDash^+_{lp} \Delta$ iff $\Gamma \vdash \Delta$.*

Proof. The soundness proof is straightforward. The completeness proof covered by Priest (2008)[pg 157; Thm 8.7.9] for the corresponding tableau system carries over directly, where, as above, the negative tableau tag corresponds to the right position in our sequents, and the positive the left.[13] □

THEOREM 0.15 (Cut Elimination) *If* $\Gamma \vdash \Delta, A$ *and* $\Sigma, A \vdash \Pi$ *then* $\Gamma, \Sigma \vdash \Delta, \Pi$.

Proof. The given completeness proof is Cut-free. □

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I should also briefly note my hope that this paper be part of a bigger project. In fact, my chief interest is not so much in all of the details of LP^+ as it is in the philosophical application: my chief interest—though not in this paper—is in advancing a version of 'dialetheism' underwritten by LP^+ along the lines of what Graham Priest and I (in conversation) call the *Goodship Project* (Goodship, 1996), specifically a version of 'non-detachable dialetheism' (that lacks a detachable conditional), but this is for a larger project. In the current paper, I ony present LP^+ and a general perspective on using multiple-conclusion logics, all with the target philosophical application to the idea of 'default classicality'.

BIBLIOGRAPHY

Anderson, A. R., Belnap, N. D., & Dunn, J. (1992). *Entailment: The Logic of Relevance and Necessity*, Volume 2. Princeton: Princeton University Press.

Asenjo, F. G. (1966). A calculus of antinomies. *Notre Dame Journal of Formal Logic* **16**, 103–105.

[13] For the interested reader: the given proof (Priest, 2008, pg 157; Thm 8.7.9) is done via so-called relational semantics for LP, which uses 'exhaustive subsets' of $S \times \{1, 0\}$ for interpretations, where a subset is exhaustive just if it contains $\langle A, 1\rangle$ or $\langle \neg A, 1\rangle$ for each sentence A in S. Other (obvious) constraints on interpretations are imposed (e.g., requiring that $\langle A \wedge B, 1\rangle$ be in an interpretation just if $\langle A, 1\rangle$ and $\langle B, 1\rangle$ are in there, etc.). For purposes of the completeness proof—in particular, defining *induced interpretation*—one may translate the relational semantics into the many-valued semantics (used here) in various ways, the easiest involving two steps. Step one (Dunn, 1966): map our set of semantic values $\{1, .5, 0\}$ into $\wp(\{1, 0\})$ so that the images of 1, .5, and 0 are $\{1\}$, $\{1, 0\}$, and $\{0\}$, respectively. Step two: where $\rho \subseteq S \times \{1, 0\}$, and v is one of our LP valuations from S into (now) $\wp(\{1, 0\})$, set

- $1 \in v(A)$ iff $\langle A, 1\rangle \in \rho$;
- $0 \in v(A)$ iff $\langle A, 0\rangle \in \rho$.

The rest of the proof carries over keeping in mind the 'positive as left position' and 'negative as right position' ideas.

Baaz, M., Fermüller, C. G., & Zach, R. Dual systems of sequents and tableaux for many-valued logics. Technical Report TUW-E185.2-BFZ.2-92.

Baaz, M., Fermüller, C. G., & Zach, R. Systematic construction of natural deduction systems for many-valued logics: extended report. Technical Report TUW-E185.2-BFZ.1-93.

Beall, J. (2009). *Spandrels of Truth*. Oxford: Oxford University Press.

Beall, J. (2010). *Logic: The Basics*. Oxford: Routledge.

Beall, J., & Ripley, D. (2011). *Supplement to Logic: The Basics*. Oxford: Routledge. Available at `entailments.net`.

Beall, J., & van Fraassen, B. C. (2003). *Possibilities and Paradox: An Introduction to Modal and Many-Valued Logic*. Oxford: Oxford University Press.

Belnap, N. D., & Dunn, J. M. (1973). Entailment and the disjunctive syllogism. In Fløistad, F. & von Wright, G. H., editors, *Philosophy of Language/Philosophical Logic*, pp. 337–366. The Hague: Martinus Nijhoff. Reprinted in (Anderson et al., 1992, §80).

Brady, R. (2006). *Universal Logic*, Volume 109. Stanford, CA: CSLI Lecture Notes.

Carroll, L. (1895). What the Tortoise said to Achilles. *Mind* **4**(14), 278–280.

Dunn, J. M. (1966). *The Algebra of Intensional Logics*. Ph. D. thesis, University of Pittsburgh.

Field, H. (2008). *Saving Truth from Paradox*. Oxford: Oxford University Press.

Field, H. (2010). Reply to Restall. *Philosophical Studies* **147**, 460–467. This is part of a book symposium on Field (2008).

Goodship, L. (1996). On dialetheism. *Australasian Journal of Philosophy* **74**, 153–161.

Harman, G. (1986). *Change in View: Principles of Reasoning*. Cambridge, MA: MIT Press.

Kripke, S. (1975). Outline of a theory of truth. *Journal of Philosophy* **72**, 690–716. Reprinted in Martin (1984).

Martin, R. L., editor (1984). *Recent Essays on Truth and the Liar Paradox*. New York: Oxford University Press.

Priest, G. (1979). The logic of paradox. *Journal of Philosophical Logic* **8**, 219–241.

Priest, G. (1984). Logic of paradox revisited. *Journal of Philosophical Logic* **13**, 153–179.

Priest, G. (1991). Minimally inconsistent LP. *Studia Logica* **50**, 321–331.

Priest, G. (2006a). *Doubt Truth To Be A Liar*. Oxford: Oxford University Press.

Priest, G. (2006b). *In Contradiction* (Second ed.). Oxford: Oxford University Press. First printed by Martinus Nijhoff in 1987.

Priest, G. (2008). *An Introduction to Non-Classical Logic* (Second ed.). Cambridge: Cambridge University Press. First edition published in 2001.

Restall, G. (2004). Laws of non-contradiction, laws of the excluded middle and logics. In Priest, G., Beall, J., & Armour-Garb, B., editors, *The Law of Non-Contradiction*, pp. 73–85. Oxford: Oxford University Press.

Restall, G. (2005a). *Logic: An Introduction*. New York: Routledge.

Restall, G. (2005b). Multiple conclusions. In Hájek, P., Valdes-Villanueva, L., & Westerstahl, D., editors, *Logic, Methodology and Philosophy of Science: Proceedings of the Twelth International Congress*, pp. 189–205. King's College Publications.

Ripley, D. (2011). Conservatively extending classical logic with transparent truth. To appear. Presented at a 2010 meeting of the Melbourne Logic Group.

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