# Nonclassical theories of truth

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This chapter attempts to give a brief overview of *nonclassical* (-logic) theories of truth. Due to space limitations, we follow a victory-through-sacrifice policy: sacrifice details in exchange for clarity of big-picture ideas. This policy results in our giving all-too-brief treatment to certain topics that have dominated discussion in the non-classical-logic area of truth studies. (This is particularly so of the 'suitable conditoinal' issue: §4.3.) Still, we present enough representative ideas that one may fruitfully turn from this essay to the more-detailed cited works for further study. Throughout – again, due to space – we focus only on the most central motivation for standard non-classical-logic-based truth theories: namely, truth-theoretic paradox (specifically, due to space, the liar paradox).

Our discussion is structured as follows. We first set some terminology concerning theories and logics; this terminology allows us to frame the discussion in a broad-but-clean fashion. (On the logic side, we present a very basic sequent system for truth and negation – and nothing more.) We then present a stripped-down version of the liar paradox. The paradox, as we set it up, turns on four basic rules (not including the truth rules; it's the job of our target non-classical truth theories to preserve these in unrestricted form): two rules governing negation's behavior, and two rules governing the 'structure' of the validity relation itself. These four rules serve as choice points for the four basic theoretical directions that we sketch. While details, as warned above, are sacrificed for space and big-picture clarity, we hope that the discussion nonetheless charts the main directions of non-classical response to basic truth-theoretic paradox.

## 1 Theories and logics

Since we'll be considering a variety of logics in this chapter, it will help to first have some tools to work with. We'll adapt, and slightly broaden, the framework of [Restall, 2005] to this end. For purposes of framing our discussion, we take a *theory* to be a record of both what the given theorist – one who endorses the given theory – accepts and what she rejects (with respect to the given phenomena). Hence, we shall take a theory $\mathcal{T}$ to be a pair $\langle \mathcal{A}, \mathcal{R} \rangle$, where $\mathcal{A}$ and $\mathcal{R}$ contain what an endorser of $\mathcal{T}$ accepts and rejects, respectively.

For some kinds of theory, we might be able to figure out what must be in $\mathcal{R}$ by looking at $\mathcal{A}$ (e.g., each negation in $\mathcal{A}$ might correspond 1-1 to an entry in $\mathcal{R}$), or vice versa. This is the usual situation with classical theories and classical logic: a classical theorist rejects something iff she accepts its negation. We shall look at two theories that have this feature (see §5.1 and §5.2). On the other hand, some theories may lack this feature: it might be that neither $\mathcal{A}$ nor $\mathcal{R}$

provides sufficient information to derive the other (e.g., negation might fail to track rejection). We shall look at two theories that have this feature (see §4.1 and §4.2).

Each sort of theory we discuss comes with a particular logical approach. We take logics to constrain theories as follows, again following [Restall, 2005]. The argument from premises $\Gamma$ to conclusions $\Delta$ is valid (we write $\Gamma \vdash \Delta$) iff it's out of logical bounds to adopt a theory $\langle \mathcal{A}, \mathcal{R} \rangle$ such that $\Gamma \subseteq \mathcal{A}$ and $\Delta \subseteq \mathcal{R}$. In short: a valid argument rules out certain theories, notably, those theories that accept all of the (valid) argument's premises and reject all of its conclusions.

Finally, the logics that we discuss all exhibit two familiar features:

- reflexivity: $A \vdash A$, for any claim $A$.
- monotonicity: let $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. If $\Gamma \vdash \Delta$, then $\Gamma' \vdash \Delta'$.

In terms of the interplay with theories, reflexivity tells us that no (logically acceptable) theory $\langle \mathcal{A}, \mathcal{R} \rangle$ involves overlap: $\mathcal{A} \cap \mathcal{R} = \emptyset$. In other words, logic, being reflexive, forbids theorists from both accepting and rejecting one and the same thing.[1] For monotonicity, define a $\mathcal{T}$*-expanded theory* to be any theory $\mathcal{T}' = \langle \mathcal{A}', \mathcal{R}' \rangle$ achieved via superset: $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{R} \subseteq \mathcal{R}'$. Then monotonicity tells us that if (the given) logic rules out a theory $\mathcal{T}$, it rules out every $\mathcal{T}$-expanded theory too. In other words, if logic rules out accepting $\Gamma$ while rejecting $\Delta$, then adding more acceptances or rejections won't help.[2]

# 2 Reasoning with truth

Throughout the paper, we use $T$ as our truth predicate, and take $\langle A \rangle$ to be a singular term referring to the sentence $A$. We simply assume that each sentence $A$ has some such name $\langle A \rangle$, without fussing about *how* $\langle A \rangle$ comes to refer to $A$; it can be a quote name, a proper name, a definite description, a Gödel code, or whatever.

There are various familiar principles of reasoning relating $A$ to $T\langle A \rangle$; we consider three candidates: transparency, the $T$-schema, and capture and release.

### Transparency

*Transparency* is the principle that $A$ and $T\langle A \rangle$ are intersubstitutable with each other in all non-opaque contexts. Ignoring opaque contexts, transparency amounts to everywhere-intersubstitutability. This requires not only that $A$ be equivalent to $T\langle A \rangle$, but also that $A \wedge (\neg B \supset T\langle C \rangle)$ be equivalent to $T\langle T\langle A \rangle\rangle \wedge T\langle \neg T\langle B \rangle \supset C \rangle$, and so on. In short, $T$s can be added and subtracted willy-nilly, to whole formulas or subformulas. Let formulas that can be obtained from each other by adding and subtracting $T$s be called $T$*-variants*.

The notion of equivalence in play can be specified in a few ways. As a constraint on theories, the most natural understanding is this: a theory $\langle \mathcal{A}, \mathcal{R} \rangle$ obeys transparency iff for all $A$, if $A \in \mathcal{A}$ then every $T$-variant of $A$ is in $\mathcal{A}$ as well; and if $A \in \mathcal{R}$ then every $T$-variant of $A$ is in $\mathcal{R}$ as well. This results in

[1] For an approach to paradox that does without this constraint, see [Ripley, 2011].

[2] For convenience, we speak of *accepting (set)* $\Gamma$ and *rejecting (set)* $\Delta$, whereby – note well – we mean *accepting everything in* $\Gamma$ and *rejecting everything in* $\Delta$, respectively.

$A$'s and all its $T$-variants being equivalent in argument: swapping formulas for their $T$-variants never makes a valid argument invalid or vice versa.

#### The $T$-schema

*The $T$-schema* is the schema $A \equiv_x T\langle A\rangle$, where $\equiv_x$ is some biconditional or other – typically, in the first instance, a material biconditional (built from negation, disjunction, and conjunction in the usual way). [Tarski, 1944] offers this schema – in material-biconditional form – as a necessary condition on theories of truth: an adequate theory, he supposes, must have every instance of the $T$-schema as a theorem. On our theory-directed interpretation of theoremhood, this amounts to the following: that a theory must not reject any instances of the $T$-schema. If a theory must not reject any instances of $A \equiv_x A$, the given ($x$-version) $T$-schema follows from transparency. But not all theories accept all versions (e.g., material-conditional version) of the $T$-schema. (See §4.3.)

#### Capture and release

*Capture* and *release* are argument forms or 'rules of inference'. Capture is the rule going from $A$ to $T\langle A\rangle$, the idea being that the truth predicate 'captures' the 'content' of $A$, and release is the converse, the rule from $T\langle A\rangle$ to $A$. On our interpretation, capture – qua logical rule – rules out any theory that accepts $A$ but rejects $T\langle A\rangle$, and release rules out any theory that accepts $T\langle A\rangle$ but rejects $A$. Given that logic is reflexive (see above), capture and release follow from transparency. (And if logic were also to enjoy a 'deduction theorem', the $T$-schema follows from capture and release; however, some of the logics discussed below do not enjoy a deduction theorem. See §4.1–§4.3.)

Clearly, transparency, the $T$-schema, and capture and release have something in common, but they spell it out in different ways. The relations between them are sometimes non-obvious, and always depend on particular features of the background logic. But the core of all three ideas is that $A$ and $T\langle A\rangle$ can stand in for each other in various essential ways. In the nonclassical theories sketched below, this core idea remains fixed: at the very least, truth plays capture and release (if not also being transparent).

## 3 Paradox and classical logic

In many languages (all natural languages and some formal ones), a sentence can contain a singular term referring to that very sentence itself. For example, the sentence 'This very sentence has twenty-three words' includes the singular term 'this very sentence'; given a certain context, this term can refer to the sentence itself, rendering it false.

Our main concern in this section is a liar sentence $\lambda$ which, one way or another, just is $\neg T\langle\lambda\rangle$. In other words, $\lambda$ is a sentence that says of itself (only) that it is not true. We can produce such a thing in any number of ways, and we won't particularly worry about how the trick is pulled here.[3] The liar causes its trouble by, in some sense, being able to stand in for its own negation. (The

[3] For concreteness, we can take $\lambda$ to be the sentence 'The quoted sentence in footnote 3 is not true'.

precise sense of *standing in* depends on which properties are taken to govern the truth predicate. We shall, for space reasons, pass over exact details.)

Reasoning classically, we can see that this causes trouble as follows: we cannot reject both the liar and its negation. But since it can stand in for its own negation, this means that we cannot reject both the liar and itself; in other words, we cannot reject it. On the other hand, we cannot accept both the liar and its negation. Since it can stand in for its own negation, this means we cannot accept both the liar and itself; in other words, we cannot accept it. Trouble seems to be afoot.

The classical principles invoked in the foregoing liar-paradoxical reasoning may be summarised as follows: 1) for any sentence, we cannot reject it together with its negation; 2) for any sentence, we cannot accept it together with its negation; 3) if we cannot reject a sentence together with itself, we cannot reject the sentence; 4) if we cannot accept a sentence together with itself, we cannot accept the sentence; and 5) if we cannot accept a sentence and cannot reject it, trouble is afoot.

## 3.1 The liar in sequent form

We proceed to make the given liar-paradoxical argument precise via a Gentzen-style sequent calculus. For our purposes, we needn't worry about conjunction, disjunction, a conditional, quantifiers, or any of that; the rules governing negation, along with the so-called *structural rules*, suffice to cause trouble. (We thus won't consider approaches, like supervaluational or subvaluational approaches, that hinge on fiddling with the behavior of conjunction and disjunction. See [McGee, 1991, van Fraassen, 1968, van Fraassen, 1970].)

Our sequents are things of the form $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are finite 'multisets' of formulas. A *multiset* is just like a set, except things can be members of it multiple times, and it matters how many times something is a member [Meyer and McRobbie, 1982a, Meyer and McRobbie, 1982b]. Thus, the multiset $[A, A]$ is different from the multiset $[A]$, even though the set $\{A, A\}$ is the same set as $\{A\}$. Multisets do *not* pay attention to order; thus, the multiset $[A, B]$ is the same multiset as $[B, A]$. In an argument with multiple premises, the premises are (as usual) interpreted *conjunctively*; multiple conclusions are dually interpreted *disjunctively*.

In our simple Gentzen system (our *logic*), we take as axioms all sequents of the form $\Gamma, A \vdash A, \Delta$,[4] and proceed to add three kinds of rules: *contraction* rules, a *cut* rule, and *negation* rules. The first two kinds are *structural*: they don't involve any particular vocabulary. The last kind is *operational*: it tells us what rules negation obeys. First, the contraction rules:

| | | | |
|---|---|---|---|
| Contraction L: | $\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$ | Contraction R: | $\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$ |

**Figure 1:** Contraction rules

These tell us that whenever we have multiple occurrences of a premise or a

[4]We set up our axioms with side premises $\Gamma$ and side conclusions $\Delta$ so that all the logics we consider will be *monotonic*: adding premises or conclusions can never make a valid argument invalid. Monotonicity does not seem to be implicated in any of the paradoxes of truth, so we hold it innocent here.

conclusion in a valid argument, the argument remains valid with just a single occurrence of that premise or conclusion. They preserve classical validity, and indeed play a key role in sequent calculi for classical logic. In terms of theories, they tell us that accepting or rejecting something twice is no stronger than accepting or rejecting it once.

In addition to the two contraction rules (both structural rules), our liar-paradoxical reasoning also includes the following structural rule:

$$\text{Cut:} \quad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

**Figure 2:** Cut rule

Cut encodes the *transitivity* of our consequence relation: if $B$ entails $A$ and $A$ entails $C$, then the cut rule guarantees that $B$ entails $C$ directly; the formula $A$ can be cut out, and argument may proceed directly from $B$ to $C$. Cut also preserves classical validity in the usual presentations. Unlike the contraction rules, however, the rule of cut is typically *eliminable*; that is, it does not expand the stock of provable sequents. It merely provides shortcuts, allowing smaller derivations of some of the very same sequents. In terms of theories, cut is an extensibility condition: it tells us that if some commitments rule out rejecting $A$, and other commitments rule out accepting it, then combining all of those commitments is ruled out. A theory doesn't have to actually take a stand on $A$; cut requires each theory to at least leave open some stand on $A$.

Finally, our liar-paradoxical argument depends on *operational* rules, namely, rules governing the operator negation. We use the usual classical negation rules:

$$\neg\text{L:} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad\qquad \neg\text{R:} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

**Figure 3:** Negation rules

These rules encode the *flip-flop* behavior of classical negation. From the axiom $A \vdash A$, they allow us to prove pivotal sequents:

- *exclusion*: $A, \neg A \vdash$
- *exhaustion*: $\vdash A, \neg A$

Exclusion, derived via ¬L and reflexivity, tells us that $A$ and its negation may not be accepted together. The second, derived via ¬R and reflexivity, tells us that $A$ and its negation may not be rejected together.

The foregoing axioms and rules are part of the usual sequent presentation of classical logic; they are enough to reconstruct the above argument for liar-paradoxical trouble, at least given rules governing truth (and the existence of a liar sentence, which we assume). For present purposes, we shall work with capture and release as our rules governing truth, even though the argument can be equally reconstructed with transparency or (given rules for $\equiv_x$, for some biconditional or other) the $T$-schema. To accommodate capture and release, we take as additional axioms every instance of the following two schemas:

- capture: $\Gamma, A \vdash T\langle A\rangle, \Delta$

- release: $\Gamma, T\langle A\rangle \vdash A, \Delta$

With all of this in hand, the liar-paradoxical argument may be run as follows.

$$\dfrac{\text{Contraction R: }\dfrac{\neg\text{R: }\dfrac{p, T\langle\lambda\rangle \vdash \lambda}{p \vdash \lambda, \neg T\langle\lambda\rangle}}{p \vdash \lambda} \qquad \text{Contraction L: }\dfrac{\neg\text{L: }\dfrac{\lambda \vdash T\langle\lambda\rangle, q}{\lambda, \neg T\langle\lambda\rangle \vdash q}}{\lambda \vdash q}}{p \vdash q}\ \text{Cut}$$

(For the contraction steps, recall that $\lambda$ just is $\neg T\langle\lambda\rangle$, so we genuinely are contracting two occurrences of the same sentence.) The resulting sequent $p \vdash q$ is absurd: $p$ and $q$ are arbitrary, so a logic that delivers $p \vdash q$ is one according to which anything (whatsoever) entails anything else (whatsoever). This, for our purposes, is completely unacceptable, and so something has to go.[5] If we take the classical principles appealed to in this argument to be nonnegotiable, then it's clear where the adjustment has to be: capture and release (and transparency and the $T$-schema, as they're implicated in related versions of this argument) must be given up, and so must any theory that entails them. A theory that maintains capture and release, then, must be couched in a logic that does not accept all of ¬R, ¬L, contraction, and cut. As usual, relaxing logical principles opens space for new theories, theories that would be ruled out if stronger logical principles were held fast.

Here, we discuss four logical options in turn: 1) getting rid of ¬R; 2) getting rid of ¬L; 3) getting rid of cut; and 4) getting rid of contraction. These four logical options open up different sorts of space for a theory of truth to occupy. As part of our discussion, we also briefly sketch the sort of theory that can live in each kind of logical environment.

# 4 Operational approaches

Operational approaches are ones that target a particular operator (or class of operators) as the source of liar-paradoxical trouble. In our sample liar derivation above (see §3.1), the only operator involved is negation. The directions of operational approaches that we shall present are those that target negation as the source of trouble – at least initially. (For the potential of additional trouble arising from Curry's paradox, see §4.3.)

## 4.1 Getting rid of ¬R: paracomplete solutions

Getting rid of ¬R amounts to rejecting exhaustion; logical approaches that take this route are known as *paracomplete*. Such logics allow for *paracomplete theories*, where a theory $\mathcal{T} = \langle \mathcal{A}, \mathcal{R}\rangle$ is paracomplete just if both $B$ and $\neg B$ are in $\mathcal{R}$ for some (but not all) sentence(s) $B$. With respect to the liar, paracomplete theorists reject $\lambda$ but also reject $\neg\lambda$.

[5] Some accept the conclusion [Azzouni, 2006, Kabay, 2010], but we won't rebut their arguments here. Our goal is to sketch some of the motivations for nonclassical theories, and one such motivation is to avoid this trivialist conclusion.

#### 4.1.1 Excluded middle

Generally, provided that disjunction $\vee$ exhibits standard behavior, paracomplete theorists reject *excluded middle* in the form

$$B \vdash A \vee \neg A$$

This is not to say that paracomplete theorists reject *all instances* of $A \vee \neg A$. Such theorists might think – for extra-logical, certain theory-specific reasons – that, for some specific fragment of the language (e.g., $T$-free fragment, physics, some such), all instances of $A \vee \neg A$ hold [Field, 2008]. But they reject that $A \vee \neg A$ is logically true – holds via logic alone.

The failure of excluded middle affects the options for $T$-biconditionals in such theories. This topic is (briefly) discussed below (see §4.3).

### 4.2 Getting rid of ¬L: paraconsistent solutions

Getting rid of ¬L amounts to rejecting exclusion; logical approaches that take this route are known as *paraconsistent*. Such logics allow for *paraconsistent theories*, where a theory $\mathcal{T} = \langle \mathcal{A}, \mathcal{R} \rangle$ is paraconsistent just if both $B$ and $\neg B$ are in $\mathcal{A}$ for some (but not all) sentence(s) $B$.[6]

#### 4.2.1 Explosion

Generally, provided that conjunction $\wedge$ exhibits standard behavior, paraconsistent theorists reject *explosion* in the form

$$A \wedge \neg A \vdash B$$

This is not to say that paraconsistent theorists accept *all instances* of $A \wedge \neg A$. Such theorists might think – for extra-logical, certain theory-specific reasons – that, for some specific fragment of the language (e.g., $T$-free fragment, physics, some such), all instances of $A \wedge \neg A$ fail to hold [Beall, 2009]. But they reject that $A \wedge \neg A$ is logically untrue – fails via logic alone.

The failure of explosion affects the options for $T$-biconditionals in such theories – a topic to which we now (briefly) turn.

### 4.3 Suitable conditionals and Curry's paradox

Our given paracomplete and paraconsistent theories wind up with a non-classical material conditional, where a material conditional $A \supset B$ is defined as $\neg A \vee B$.

- Paracomplete: $\nvdash A \supset A$.
- Paraconsistent: $A, A \supset B \nvdash B$.

[6] We should note here that we shall – with reluctance – use 'paraconsistent theories' in a way that coincides with what have come to be called 'dialetheic theories'. There are important distinctions to be drawn here, but they would consume too much space for present purposes. We note that the term 'dialetheic' (similarly, 'dialetheist') is due to Priest and Routley [Priest and Routley, 1989, Priest, 2006], who, along with Mortensen [Mortensen, 1995] and others, were pioneers of such (so-called strong paraconsistent) theories. See too [Asenjo, 1966].

Hence, in either case, the resulting material conditional is often thought to be inadequate for purposes of underwriting the $T$-biconditionals.[7] In the paracomplete case, the given conditional detaches (i.e., satisfies modus ponens) but fails to support all instances of the given (material) $T$-schema: $T\langle A\rangle \supset A$ and its converse can fail. In the paraconsistent case, all instances of the given (material) $T$-schema hold; however, the given conditional fails to detach.

As a result of these apparent deficiencies, much of the work in paraconsistent and paracomplete responses to paradox has focused on supplementing such theories with a suitable conditional, one that both detaches and validates all $T$-biconditionals [Beall, 2009, Field, 2008, Priest, 2006, Brady, 2006]. But the task is difficult. What makes the task particularly difficult is Curry's paradox [Meyer et al., 1979], which involves (conditional) sentences that say of themselves (only) that *if* they are true *then* absurdity is true (e.g., that everything is true).[8] In the material-conditional case, Curry's paradox is nothing more than a disjunctive version of the liar (e.g., 'Either I'm not true or absurdity is true'), which is already treated by standard paraconsistent or paracomplete approaches to the liar. But when a new 'suitable conditional' has been added to the mix, Curry's paradox is a distinct – and very, very difficult – problem [Myhill, 1975]. In fact, Curry's paradox has often been regarded as the hardest obstacle in the path of para- solutions to paradox [Beall et al., 2006, Field, 2008, Priest, 2006].[9]

For space reasons, we need omit discussion of the various avenues towards adding detachable, but Curry-paradoxical-safe, $T$-biconditionals to paracomplete and paraconsistent theories [Beall, 2009, Brady, 2006, Field, 2008, Priest, 2006]. But we should mention a relatively unexplored alternative: simply accept the deficiencies of the material $T$-biconditionals, but respond to them in some other fashion. One approach is to devise a suitable non-monotonic

[7]One easy way to establish such 'inadequacies' is via a common (sound and complete) 'semantics' for common such logics – for example, Strong Kleene or K3 [Kleene, 1952, Beall and van Fraassen, 2003] and LP [Asenjo, 1966, Priest, 1979, Beall and van Fraassen, 2003]. In short: let $\mathbb{V}$ contain all (total) maps $v : \Sigma \longrightarrow \{1, .5, 0\}$ from sentences into $\{1, .5, 0\}$ such that $v(\neg A) = 1 - v(A)$, and $v(A \wedge B) = \min\{v(A), v(B)\}$ and $v(A \vee B) = \max\{v(A), v(B)\}$. In the paracomplete K3 case, we say that $v \in \mathbb{V}$ *satisfies* $A$ just if $v(A) = 1$, and *dissatisfies* $A$ otherwise. In the LP case, we say that $v \in \mathbb{V}$ *satisfies* $A$ just if $v(A) \in \{1, .5\}$, and *dissatisfies* $A$ otherwise. In both cases, we say that $v \in \mathbb{V}$ satisfies a set $\Gamma \subseteq \Sigma$ iff $v$ satisfies each member of $\Sigma$, and $v \in \mathbb{V}$ *dis*satisfies $\Gamma$ iff $v$ dissatisfies *all* elements of $\Gamma$. Finally, we may define, for each of the given logics $L$, 'semantic consequence' $\vdash_L$ in the foregoing terms: $\Gamma \vdash_L \Delta$ iff there's no $v \in \mathbb{V}$ that satisfies $\Gamma$ but dissatisfies $\Delta$. Where $L$ is taken to be K3, with (dis-) satisfaction defined as above, $\vdash_L$ is paracomplete (as an easy exercise shows); and, dually, $\vdash_L$ is paraconsistent where $L$ is taken to be LP, with (dis-) satisfaction defined as above. (NB: we have actually given what we have elsewhere called K3$^+$ and LP$^+$, respectively, in order to maintain uniformity with our multiple-conclusion-based discussion in sequent-calculus terms. See [Beall, 2011]. Strictly speaking, K3 and LP are the single(-ton)-conclusion limits of K3$^+$ and LP$^+$, so understood.)

[8]Worth noting here is that in popular paracomplete logics such as Strong Kleene, the material conditional fails to enjoy a deduction theorem. Example: $A \vdash A$ but $\nvdash A \supset A$. On the other (dual) side, with the corresponding (dual) paraconsistent logic LP, the other direction of the deduction theorem fails: $\vdash (A \wedge (A \supset B)) \supset B$ but $A \wedge (A \supset B) \nvdash B$. In general, for Curry-paradoxical reasons, theories cannot have a deduction theorem for a detachable conditional – at least if the underlying structural rules contain both transitivity and contraction. (See §5 for more discussion.))

[9]While we cannot discuss it, we should mention too that Curry's paradox equally confronts 'property theories' that purport to accommodate properties corresponding to each meaningful predicate – in short, each meaningful predicate picks out a property exemplified by all and only the objects of which the predicate is true. Having this sort of theory confronts Curry's paradox in the Russell-like form of '$x$ does not exemplify $x$', etc.

logic, and try to 'capture back' as much of the otherwise lost features of the $T$-biconditionals [Goodship, 1996, Priest, 1991]. Another route is to move to a multiple-conclusion logic and an appropriate philosophy thereof (e.g., one that sees the work of 'detachment' not in a detachable conditional but instead in extralogical principles that ground the inference from certain premises to certain conclusions) [Beall, 2011]. The viability of such approaches remains open.

# 5 Substructural approaches

The above approaches work at the level of *operational* rules, in particular the rules governing negation. But classical negation is useful for many purposes. For example, as we've seen above, paracompletists and paraconsistentists alike must reject the usual understanding of the relations between acceptance, rejection, and negation: paracomplete theorists *reject* some $A$ without thereby *accepting* $\neg A$, while paraconsistent theorists *accept* some $\neg A$ without thereby rejecting $A$, and so on. In addition, the paracompletist loses the law of excluded middle, and the paraconsistentist loses explosion, both familiar and useful principles of inference. Finally, the loss of excluded middle or explosion removes much of the conditional flavor of the classical material conditional. For these reasons, an approach that allows us to proceed without losing so much might be thought superior over the para- accounts.

Here, we briefly outline two *substructural* approaches. These work at the level of the structural rules, so they allow for the maintenance of both ¬L and ¬R, restoring much of the usefulness of classical negation and the classical material conditional. But they too are not without costs, as we note below.

## 5.1 Getting rid of cut: nontransitive solutions

The first substructural approach we consider retains the rules of contraction and dispenses with the rule of cut; this results in a *nontransitive* logic. On an approach like this, both of the sequents $p \vdash \lambda$ and $\lambda \vdash q$ are derivable, but without the rule of cut there is no way to derive $p \vdash q$, so the disaster is averted at the very last step.

Nontransitive logics have been advanced in [Weir, 2005, Ripley, 2011] for handling truth-theoretic paradoxes. They block the problematic derivation, and they do so in a way that allows them to preserve classical operational rules. (The system presented in [Weir, 2005] preserves many, but not all, classical operational rules; the system presented in [Ripley, 2011] preserves them all. As a result, we focus in this section on the latter system.) This allows the resulting logical systems to behave quite naturally in a number of ways.

By preserving the classical flip-flop behavior of negation, the nontransitive theorist also preserves the conditional flavor of the material conditional. Nontransitive logics, like the logic ST discussed in [Ripley, 2011], can maintain the trinity which the para- approaches, in one way or another, abandon:

- $\supset$-identity: $\vdash A \supset A$
- $\supset$ modus ponens: $A, A \supset B \vdash B$
- deduction theorem: $\Gamma, A \vdash B, \Delta$ iff $\Gamma \vdash A \supset B, \Delta$.

In fact, approaches that focus exclusively on operational rules not only must fail some of these for the material conditional, but in fact must fail some of these for *any* connective, due to Curry paradox. (Proof: exercise, but use the above rules and standard structural rules, plus release and capture.) This means that nontransitive logics can make do with material conditionals and, in fact, material $T$-biconditionals: there is no need either to add a separate 'suitable conditional' or to learn to live with oddly-behaved conditionals – unlike in the paracomplete and paraconsistent theories, which, as mentioned in §4.3, must take one of these routes.

There is a reason why nontransitive logics can behave so classically. Recall that cut, unlike contraction, is eliminable in many presentations of (truth-free) classical logic; this means that it plays no essential role in any derivation. Anything that can be derived with it can also be derived without it. As our above liar-based argument shows, this is no longer true when the behavior of truth is accounted for; with capture and release on board, cut makes a genuine difference. However, it only makes a difference to derivations in which capture and release are involved; as a result, one can preserve *every* classically-valid argument in a nontransitive logic. As is shown in [Ripley, 2011], one can even ensure that all of these arguments extend to cover the full, truth-involving, language.

There is thus a clear sense in which such a nontransitive system is not nonclassical: it validates every classically-valid argument. Nonetheless, the loss of transitivity is at least unfamiliar, and the motivations for adopting such a logic are very similar to many nonclassicists' motivations; there is an equally clear sense in which such an approach *is* nonclassical. We won't bother with the terminological question here.

As we sketched above, the rule of cut amounts to the following constraint on theories: every theory must leave open either accepting $A$ or rejecting it. Ripley takes $\lambda$ to provide a counterexample to this principle and thus to transitivity. Deriving $\vdash \lambda$ thus tells us that it's incoherent to reject $\lambda$, and deriving $\lambda \vdash$ that it's incoherent to accept it. The nontransitivist of this stripe must neither accept nor reject $\lambda$. This is the theory offered of $\lambda$'s paradoxicality: it cannot be accepted or rejected without incoherence. Unlike the operational approaches, this nontransitive theory maintains the equivalence between accepting $\neg A$ and rejecting $A$, and between rejecting $\neg A$ and accepting $A$. Thus, $\neg\lambda$ too must be neither accepted nor rejected. In acceptance, then, this approach is like a paracomplete approach: it accepts neither $\lambda$ nor $\neg\lambda$. In rejection, it is like a paraconsistent approach: it rejects neither $\lambda$ nor $\neg\lambda$. However, given our above definitions, this theory is neither paracomplete nor paraconsistent.[10]

## 5.2 Getting rid of contraction: noncontractive solutions

The other sort of substructural approach we'll consider retains the rule of cut, and does without the rules of contraction. Such an approach is recommended and outlined in [Beall and Murzi, 2011, Shapiro, 2010, Zardini, 2011]. On a noncontractive approach, one can allow that the sequents $p \vdash \lambda, \lambda$ and $\lambda, \lambda \vdash q$ are derivable, but insist that the sequents $p \vdash \lambda$ and $\lambda \vdash q$ are not; this blocks

[10] For a variant nontransitive theory that is both paracomplete and paraconsistent on the present definitions, see [Ripley, 2011].

the derivation of $p \vdash q$.[11]

Moreover, it blocks the derivation in a way that allows for the negation rules and the cut rule to be preserved. This allows the resulting logical systems to behave quite intuitively in a number of ways. By preserving the classical 'flip-flop' behavior of negation, the noncontractive theorist, like the nontransitive theorist, preserves the conditional flavor of the material conditional. Noncontractive logics can thus also maintain all of $\supset$-identity, modus ponens, and the deduction theorem.[12]

If $\wedge$ is the conjunction that reflects the operation of premise combination (multiplicative conjunction (see footnote 12)), then it is no longer idempotent on a noncontractive logic; $A \wedge A$ is stronger than $A$ alone. Similarly, if $\vee$ is the disjunction that reflects the operation of conclusion combination (multiplicative disjunction (see footnote 12)), then it too is no longer idempotent; $A \vee A$ is weaker than $A$ alone. It is these differences that are exploited in the noncontractive approach to paradoxes. By arguments similar to those in §3.1, we have both $\vdash \lambda, \lambda$ and $\lambda, \lambda \vdash$ without any uses of contraction. If $\wedge$ and $\vee$ are as above, this means we have $\vdash \lambda \vee \lambda$ and $\lambda \wedge \lambda \vdash$; that is, $\lambda \vee \lambda$ is a logical truth, and $\lambda \wedge \lambda$ is explosive. Classically, this would be a problem, since classically $A \vee A$ is equivalent to $A \wedge A$. But noncontractively this is not so; since $\lambda \vee \lambda$ is weaker than $\lambda \wedge \lambda$, this is no trouble at all.

The noncontractive approach requires us to add subtlety to our account of theories. Recall that for the other approaches we consider, a theory is a pair of *sets*: $\mathcal{A}$, the things accepted by the theory, and $\mathcal{R}$, the things rejected by the theory. We then said that $\Gamma \vdash \Delta$ iff it's ruled out to accept everything in $\Gamma$ and reject everything in $\Delta$. In a noncontractive logic, however, we can have $\Gamma \vdash A, A, \Delta$ without $\Gamma \vdash A, \Delta$: it can be that rejecting $A$ twice is ruled out but rejecting $A$ once is not. This means that, to specify a theory in a noncontractive logic, we need to keep track of more than *whether* something is accepted or rejected; we also need to keep track of *how many times* it is accepted or rejected.

We do this as follows: a theory is still a pair $\langle \mathcal{A}, \mathcal{R} \rangle$. Now, however, $\mathcal{A}$ and $\mathcal{R}$ are no longer sets; they are rather $\omega$-long *sequences* of sets. We index them with natural numbers for easy reference: thus, $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots \rangle$, and $\mathcal{R} = \langle \mathcal{R}_1, \mathcal{R}_2, \ldots \rangle$. For any $n$, $\mathcal{A}_n$ is the set of formulas that the theory in question accepts at least $n$ times, and $\mathcal{R}_n$ is the set of formulas that the theory in question rejects at least $n$ times. Given this setup, we have $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots$ and $\mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \ldots$. Now, we can extend our reading of logical consequence to noncontractive approaches. We say that $\Gamma \vdash \Delta$ iff no theory can accept each thing in $\Gamma$ as many times as it appears in $\Gamma$ and reject each thing in $\Delta$ as many times as it appears in $\Delta$.[13]

[11] If we try to use the rule of cut to combine $p \vdash \lambda, \lambda$ and $\lambda, \lambda \vdash q$, we can only cut out a single occurrence of $\lambda$ from each sequent; we end up with $p, \lambda \vdash \lambda, q$. This is no problem; in fact, it's an axiom!

[12] Whether $\supset$-contraction is preserved depends on the precise rules used to govern $\supset$. In the absence of contraction, conjunction, disjunction, and the conditional come in two distinct flavors each; these are sometimes called 'additive' and 'multiplicative' flavors. (In the presence of both monotonicity and contraction, these two flavors are equivalent.) Noncontractive approaches retain $\supset$-contraction for the additive $\supset$, but not the multiplicative.

[13] Since $\vdash$ is still reflexive and monotonic, we have it that no $\mathcal{A}_i$ can overlap any $\mathcal{R}_j$; accepting something any number of times rules out rejecting it any number of times, and vice versa.

Since the noncontractive approach maintains that $p \vdash \lambda, \lambda$, we have it that no theory can accept $p$ even once and reject $\lambda$ twice; however, since $p \not\vdash \lambda$,[14] it's ok for a theory to accept $p$ once and reject $\lambda$ once. Similarly, since $\lambda, \lambda \vdash q$, no theory can accept $\lambda$ twice and reject $q$, but since $\lambda \not\vdash q$, it's ok for a theory to accept $\lambda$ once and reject $q$. Thus, the noncontractivist, on this reading, maintains that it's ok for a theory to accept $\lambda$ and ok for a theory to reject it, so long as it only does one of the two, and only does it once. The natural question at this point is: how can it be that accepting or rejecting something once can be ok when accepting or rejecting it twice is out of bounds?

[Zardini, 2011] suggests that the liar sentence exhibits a kind of *instability* reminiscent in some ways of a revision theory (see elsewhere in this volume, presumably). The idea is that from a single occurrence of $\lambda$ one may derive (via the truth rules) $\neg\lambda$, but in the process of doing this the original occurrence of $\lambda$ was destroyed; thus, we don't have $\lambda$ and $\neg\lambda$ together, which is a good thing, since $A, \neg A \vdash$. On the other hand, if we have two occurrences of $\lambda$, we can use one to derive $\neg\lambda$. This may destroy it, but we still have another copy; we then have both $\lambda$ and $\neg\lambda$ together, which is unacceptable. This is why two occurrences of $\lambda$ are unacceptable, even though one occurrence is not. (A parallel story is to be told about why two occurrences of $\lambda$ are unrejectable, even though a single occurrence is not.) Similarly, [Beall and Murzi, 2011] suggest thinking of premises as *resources* to be drawn on in the course of a proof. If drawing on a premise uses it up, then again we can see why two occurrences can get us farther than one.

This is all reasonably hand-wavey still, but we don't doubt it can be made precise. We leave the details for another occasion.

# 6 Conclusion

Classical logic (including cut) seems to rule out the possibility of giving a theory of truth that validates capture and release, or transparency, or the $T$-schema. In this chapter, we've looked at four ways to modify this logical background to open up space for such a theory of truth, and looked at the kinds of theory that fit most naturally with each modification. Two of the modifications were to the classical theory of negation; these paracomplete and paraconsistent approaches removed the requirements of exhaustiveness and exclusiveness, respectively. Relaxing exhaustiveness allows for rejecting both the liar and its negation; relaxing exclusiveness allows for accepting them both. Changing the theory of negation has effects on the theory of the material conditional as well, and these effects are a central focus of paracomplete and paraconsistent approaches (see §4.3).

The other two modifications were to structural rules; noncontractive and nontransitive approaches can keep the full classical theory of negation, but must make adjustments elsewhere, either by supposing that two occurrences of the same premise or conclusion amount to more than a single occurrence, or else

[14] The noncontractive theorist had better not accept $p \vdash \lambda$, since then two cuts with the derivable sequent $\lambda, \lambda \vdash q$ would yield the unacceptable $p \vdash q$. Similarly, they had better not accept $\lambda \vdash q$, since then two cuts with the derivable sequent $p \vdash \lambda, \lambda$ would again yield the unacceptable $p \vdash q$. This is why the noncontractive approach quite crucially must go without *both* contraction on the left and contraction on the right; this contrasts with the operational approaches above, which only need to go without a single negation rule each, and can keep the other.

by supposing that logical consequence is nontransitive. Either way, these substructural solutions owe a theory of logical consequence that can make sense of these adjustments; we've tried to sketch what such theories might look like.

Our discussion, for space reasons, has skipped over philosophical arguments for maintaining (unrestricted) capture and release, and also skipped over topics (and common terminology) of 'gaps', 'gluts', and more. These topics are all important, and skipped here only for space reasons. We leave such topics to other discussion [Beall and Glanzberg, 2008], including much of the work we've cited throughout.

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