Conditionals, counterexamples, and ternary semantics

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- There's an intuitive condition on—or desideratum for—worlds-looking conditionals of logical-law-type strength: they ought to be untrue (or false, in some sense) at a point in the universe only if there's a *counterexample*—only if there's some point in the universe at which the antecedent is true but consequent untrue (or, again, false in some sense).
- One conspicuous problem with standard ternary semantics for logical-law-type conditionals (e.g., conditionals in B vicinity or stronger) is that they seem to buck the given counterexample condition.
- I agree that the standard ternary semantics appears to buck the given condition; however, I suggest that appearances are misleading.
- I suggest that there's a natural way to look at the semantics such that the counterexample condition is respected. What we have to do, I suggest, is simply broaden our perspective of the universe and its available points.
- (Also notable is that the proposed perspective answers another common complaint in the area—one from Ed Mares concerning the logic-fiction account of abnormal worlds, due to Priest—but I omit this here.)

1 Ternary (simplified) semantics

I give only the ideas essential for background. I skip a lot (including negation, definition of validity, definition of R^3 at normal points, and more).

- Divide our universe W into $N \neq \emptyset$ and $W \setminus N$.
- Give uniform conditions across the universe for conjunction and disjunction (and negation, but we ignore that here).[1]
  - For all $x \in W$, $x \models \alpha \wedge \beta$ if and only if $x \models \alpha$ and $x \models \beta$.
  - For all $x \in W$, $x \models \alpha \vee \beta$ if and only if $x \models \alpha$ or $x \models \beta$.

*This paper/talk spells out an idea briefly aired in *Spandrels of Truth* (Ch. 2). The idea is also discussed in a broader, Australasia-based, 10-authored joint work on the ternary relation.

[1] I assume a suitable Star treatment of negation, and so skip over falsity conditions.

- Give non-uniform conditions for the arrow:
  - For all $x \in N$, $x \models \alpha \rightarrow \beta$ *iff* for any $y \in W$, if $y \models \alpha$ then $y \models \beta$.
  - For all $x \in W \setminus N$, $x \models \alpha \rightarrow \beta$ *iff* for any $y, z \in W$ such that $Rxyz$, if $y \models \alpha$ then $z \models \beta$.

2 Counterexample constraint

The intuitive constraint—the *counterexample condition*—tells us that if a logical-law-strength conditional is untrue at a point x, then there is a counterexample: some point y in the universe at which the antecedent is true but the consequent untrue (or false in some sense).

Note that this constraint is apparently bucked in the given semantics. In particular, we can have (abnormal, though no normal) points x in the universe at which conditionals are untrue (or false in some sense) even though there's apparently no counterexample:[2]

- $x \not\models \alpha \wedge \beta \rightarrow \alpha$
- $x \not\models \alpha \rightarrow \alpha \vee \beta$
- $x \not\models \alpha \wedge (\alpha \rightarrow \beta) \rightarrow \beta$
- $x \not\models \alpha \rightarrow \alpha$
- ... and much more!

For example: let x be in $W \setminus N$ and $Rxyz$ and $y \models \alpha \wedge \beta$ but $z \not\models \alpha$. Then $x \not\models \alpha \wedge \beta \rightarrow \beta$, even though there's apparently no point u that serves as a counterexample—no u such that $u \models \alpha \wedge \beta$ but $u \not\models \alpha$. After all, conjunctions are uniform across the whole universe. (See §1.)

If we want to respect the counterexample condition (understood as above), we need to add some new primitive points (and a corresponding truth-at and false/untrue-at relations) or try to find something in the universe that already does the trick. I pursue the latter route here.

3 Letting the scales fall from our eyes

I suggest that there's a natural way of thinking about the 'ternary' semantics that respects the counterexample condition. We need to expand our perspective of the universe a bit, but do so in a way driven by a natural view of the 'ternary' relation.

[2]My interest here is not in defending the need for such points (and the corresponding logics that result), but only in noting the apparent lack of counterexample. See *Spandrels of Truth* or Priest's *Doubt Truth to be a Liar* for defense of such points.

3.1 Ternary as binary

It is not uncommon to think of all n-ary relations, for $n \geq 2$, as a binary relations. In fact, a common way of defining n-tuples in set theory is as follows.

- $\langle x \rangle$ is x.
- $\langle x, y \rangle$ is $\{\{x\}, \{x, y\}\}$.
- Triples $\langle x, y, z \rangle$ are pairs: $\langle x, \langle y, z \rangle \rangle$.
- Quadruples $\langle w, x, y, z \rangle$ are pairs: $\langle w, \langle x, y, z \rangle \rangle$.
- In general, n-tuples are pairs where the second coordinate is an $n-1$-tuple: $\langle x_1, \langle x_1, \ldots, x_{n-1} \rangle \rangle$.[3]

I suggest that we simply see our 'ternary' relation in this light: it is a binary relation, and in particular a *binary access relation*—relating a point x in the universe with all *x-accessible points*, the points relevant to the truth/falsity of the connective in question (in this case, our logical-law-strength conditional).

3.2 Half worlds and duo worlds

Let us define *half worlds* and *duo worlds* as follows.

Definition 1 (Half worlds) $\langle x, y \rangle$ is a *half world* iff $x \in W$ and $y \in W$ and $x = y$.

Definition 2 (Duo worlds) $\langle x, y \rangle$ is a *duo world* iff $x \in W$ and $y \in W$ and $x \neq y$.

With such worlds so understood, let us define a *truth-* and *falsity-at* relations for them, letting $\models_1$ be the former and $\models_0$ the latter. We do this uniformly for all sentences α.

Definition 3 (Truth at half/duo worlds) $\langle x, y \rangle \models_1 \alpha$ iff $x \models \alpha$.

Definition 4 (Falsity at half/duo worlds) $\langle x, y \rangle \models_0 \alpha$ iff $y \not\models \alpha$.

Observe that our new truth-at and falsity-at relations collapse into our original truth/untruth-at relations for all *half worlds*. In short: for any $x \in W$, we have

$$\langle x, x \rangle \models_1 \alpha \ \text{ iff } \ x \models \alpha$$

and, likewise,

$$\langle x, x \rangle \models_0 \alpha \ \text{ iff } \ x \not\models \alpha$$

[3]Quite often, the order here is reversed, but that's because we're often more interested in functions than relations generally (and, so, we want n-ary functions to be binary relations that take the left coordinate—the given n-tuple—as argument and spit out the right coordinate). But, for now, we put no special weight on functions—which makes sense in an often non-functional ternary-semantics setting. (NB: the standard criterion is satisfied: $\langle v, x \rangle = \langle y, z \rangle$ iff $v = y$ and $x = z$.)

The difference is that our new truth-/falsity-at relations allow for a sort of 'inconsistency' when we're dealing with duo worlds. In particular, while there are no *half worlds* $z = \langle x, x \rangle$ at which $\alpha \wedge \beta$ is true but α false, we can now have *duo points* $z = \langle x, y \rangle$ where this happens. We can have, for example, the following for some $z \in W \times W$.

- $z \models_1 \alpha \wedge \beta$ but $z \models_0 \beta$.
- $z \models_1 \alpha$ but $z \models_0 \alpha \vee \beta$.
- $z \models_1 \alpha$ but $z \models_0 \beta$.

And these duo worlds, I suggest, are what we were overlooking in charging that standard 'ternary' semantics buck the counterexample condition.

4 The picture broadly sketched

The proposal, in a nutshell, is that we look at what's going on as follows.

- The 'ternary' relation is really just a binary access relation: it picks out all *accessible worlds*.
- Normality is a sort of blindness: *normal worlds* see only half the story; they see only half worlds—they see all of them, but only the half ones.[4]
- Abnormality removes the scales: *abnormal worlds* see *all* worlds—all *half* worlds, but also all *duo worlds*.
- We can—I think—redo the 'ternary' semantics in terms of half and duo worlds, taking our universe to be W^2. (We then stipulate, as above, that normality is blindness, seeing only half worlds; and abnormality sees more—all accessible worlds, some of which may be duo worlds.)
- . . . Und Bob ist dein Onkel (as the German–Australians say).

5 Leftover directions – for enthusiasts or experts

Adding negation needn't force a change to the proposed picture, but it's interesting to wonder about the Star. It'd be nice to have

$$\langle x, y \rangle \models_1 \neg\alpha \text{ iff } \langle x, y \rangle \models_0 \alpha$$

but, in fact, we'll have this only if $y = x^\star$. (Hence, the only half worlds for which this holds are classical worlds.)

[4]By 'see', I mean *directly see*. (A normal world can see an abnormal one that, in turn, sees duo ones; and so the normal world might, in some indirect sense, 'see' the end of the chain, so to speak. I ignore this here, as it doesn't really bear on what I say.)