Truth, Necessity, and Abnormal Worlds

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~ AAP @ Melbourne ~
July 8 2009
Introduction

- Background: philosophy and truth theory (viz., BXTT) *Spandrels of Truth* (OUP, 2009), sometimes ‘SoT’.
- Current Aim: adding a plausible (broad) necessity operator to BXTT.
- There’s quite a bit of background (e.g., everything in SoT!) that I’ll quickly sketch!
- I will be skirting many issues along the way.
LIAR PARADOX

- Dialetheic solution: *gluts*.
- A glut is a truth with a true negation.
- Paraconsistency: $A, \neg A \not\vdash B$.

Dialetheists can differ widely on the logic of truth and reach of ‘true contradictions’. (I advocate a very, very conservative dialetheic position in *SoT*, though close to Priest’s logical framework.)

For present purposes, I will speak (contrary to fact) as if dialetheists all endorse the same basic theory/logic.
**Problem: No Suitable Conditional!**

Modus Ponens (MPP)—strictly, Material Modus Ponens (MMP)—is invalid.

- The material ‘conditional’ is a disguised disjunction: \( \neg A \lor B \). (The disguise: \( A \supset B \).)
- Suppose that \( A \lor B \) is true iff \( A \) is true or \( B \) is true (or both).
- Suppose, now, that \( A \) and \( \neg A \) are true.
- \( B \) itself needn’t be true.

So, gluts present counterexamples to MMP. We need a suitable—detachable—conditional for our truth theory (e.g., T-biconditionals)!
SOLUTION: WORLDS AND PRIMITIVE CONDITIONAL

Introduce (or acknowledge) a collection $\mathcal{W}$ of ‘worlds’ and, in turn, a primitive conditional $\rightarrow$ which is all-worlds-looking.

$$w \models A \rightarrow B \text{ iff } w' \models B \text{ if } w' \models A, \text{ for all } w' \in \mathcal{W}$$

In short: for any world $w$, our new conditional $A \rightarrow B$ is true at $w$ iff there’s no world at which $A$ is true but $B$ not.
SOLUTION: WORLDS AND PRIMITIVE CONDITIONAL

- Our models contain a set of worlds $\mathcal{W}$.
- Disjunction, Conjunction get expected truth-at-a-world conditions (and falsity- too if need be).
- Negation gets truth-at-a-world conditions that allow for gluts (but, for current purposes, no gaps).
- Validity is as usual: absence of a world (in a model) that ‘makes true’ the premises but fails to ‘make true’ the conclusion.
The chief problem with our new all-worlds-looking conditional (in an ‘all-worlds access’ universe) is that it validates *Pseudo Modus Ponens*.

\[ \text{PMP. } \vdash A \land (A \rightarrow B) \rightarrow B \]

Curry paradox combines with PMP and our T-biconditionals to generate triviality – absurdity.
Let $C$ be a Curry sentence that says $C \rightarrow \bot$ (e.g., ‘If I am true, everything is true’), so that our T-biconditional – dropping $Tr(x)$ for simplicity – gives us $C \leftrightarrow (C \rightarrow \bot)$.

1. $C \leftrightarrow (C \rightarrow \bot)$ [T-biconditional]
2. $C \land (C \rightarrow \bot) \rightarrow \bot$ [PMP]
3. $C \land C \rightarrow \bot$ [2, substitution]
4. $C \rightarrow \bot$ [3, features of $\land$]
5. $C$ [1,4 MPP]
6. $\bot$ [4,5 MPP]

So, we need to avoid PMP!
SOLUTION: ABNORMAL WORLDS AND JUMPY CONDITIONAL

Taking a page from Kripke: introduce ‘abnormal worlds’ and allow our conditional to be ‘jumpy’ between types of worlds!

- Our models acknowledge a non-empty set $N \subseteq W$ of ‘normal worlds’.
- Define all (Boolean or standard first-order) connectives uniformly over all worlds.
- For our conditional: acknowledge ‘jumpy’ behavior, with $A \rightarrow B$ behaving one way at normal points and another way at abnormal points.
JUMPY CONDITIONAL: EXAMPLE

For all normal worlds \( w \in \mathcal{N} \):

\[
    w \models A \rightarrow B \text{ iff } w' \models B \text{ if } w' \models A, \text{ for all } w' \in \mathcal{W}
\]

For all abnormal worlds \( w \in \mathcal{W} \setminus \mathcal{N} \):

\[
    w \models A \rightarrow B \text{ iff } \ldots [\text{fill in favorite account (say, arbitrary)}]
\]

* On these non-normal-worlds semantics, we define validity only over (all) normal worlds of all models.
**JUMPY CONDITIONAL: EXAMPLE**

With this setup, we keep MPP but, as wanted, lose PMP.

- **MPP:** validity is defined over normal worlds. For any normal world, \( A \rightarrow B \) is true iff there’s no \( x \in \mathcal{W} \) at which \( A \) but not \( B \) is true. Hence, for any normal world \( w \), if we’ve got \( w \models A \) and \( w \models A \rightarrow B \), we’ll have \( w \models B \).

- **No PMP!!** For abnormal worlds, we’re treating the status of \( A \rightarrow B \) in an arbitrary (or unrestricted ternary) fashion. So, just let \( \mathcal{W} = \{x, y\} \) with \( \mathcal{N} = \{x\} \), and let \( y \models A \) and \( y \models A \rightarrow B \) but \( y \not\models B \). Then \( x \not\models (A \land (A \rightarrow B)) \rightarrow B \) as there’s a point \( y \) at which \( A \land (A \rightarrow B) \) is true but \( B \) not.

(NB: the Routley–Meyer ternary relation gives a slightly less ‘arbitrary’ feel to things, but skip this topic here.)
RECAP AND MAIN ISSUE

▶ Liars motivate gluts.
▶ Gluts undermine MMP, and so push for a detachable conditional.
▶ Worlds and primitive all-worlds-looking conditional gives MPP.
▶ PMP and Curry paradox require abnormal worlds and ‘jumpy’ conditional.
▶ … in the end, we have good (transparent) truth theories in this setup (and, thanks Aussie logicians, non-triviality proofs for many such theories).
Recap and Main Issue

The main issue concerns \textit{necessity} (e.g., broad necessary truth).

- Now that we have worlds for our given truth theory, how are we to understand \textit{necessary truth}?
- The task: add a plausible, broad-necessity Box to the picture (specifically, to the target theories).

One would think that this is straightforward, but there are some surprising obstacles. [NB: it \textit{is} straightforward for the \(\rightarrow\)-free theory! (See Priest RSL for basic ideas.) Our concern is the actual truth theory.] Before turning to the obstacles and an eventual solution, some desiderata should be noted.
**Desiderata**

These are minimal (though debatable) desiderata that are assumed (and that, pending argument, dialetheists ought not have to rule out).

- **Necessitation:** If $\vdash A$ then $\vdash \Box A$.
- **Box Release (rule):** $\Box A \vdash A$.
- **Diamond Capture (rule):** $A \vdash \Diamond A$ (where $\Diamond A$ is $\neg \Box \neg A$).
- **K/Distribution (rule):** $\Box (A \rightarrow B) \vdash \Box A \rightarrow \Box B$.
- **S4/KK principle:** $\Box A \vdash \Box \Box A$.
- **S5 principles:** $\Diamond A \vdash \Box \Diamond A$ or, in another form, $\Diamond \Box A \vdash \Box A$.

Of course, if some of these ‘rule’ forms can be strengthened to *axioms*, then all to the good, but we ask at least for the weaker (rule) forms.
ASSUMPTION: PRIMITIVE BOX

- Since we already have a non-triviality (analogously, consistency) result for the target theories, the best course would be having □ to be a defined connective.
- Unfortunately, I know of no viable source in the target theories in terms of which to define necessity (given the desiderata).
- (If there are ideas, please bring them up in discussion!)
- In what follows, we assume that □ is a primitive connective.
PROPOSAL 1: UNIFORM ALL-WORLDS (UAW)

Philosophers usually think of (broad) alethic necessity along ‘all worlds’ lines. This is a natural start. (The tag ‘uniform’ concerns no distinction between types of worlds—normal or abnormal.)

- We let \( \mathcal{W} \) be our collection of worlds.
- We define our \textit{uniform, all-worlds} (UAW) Box thus:

\[
    w \models \Box A \iff w' \models A \text{ for all } w' \in \mathcal{W}
\]
UAW: Problem

- Whatever other merits the UAW approach may have, we do not have Necessitation.
- In the target theories, we have $A \rightarrow A$ true at all normal worlds, and so $\vdash A \rightarrow A$.
- The problem: we do not have $A \rightarrow A$ true at all worlds, since there are abnormal worlds at which $A \rightarrow A$ is untrue.
- Hence: on the UAW approach, we have some $A$ such that $\vdash A$ but $\nvdash \Box A$.

So, given the desiderata, we need to try something else.
PROPOSAL 2: UNIFORM ALL-*normal*-WORLDS

- Given that we have abnormal worlds around, it makes sense to restrict our Box only to such normal worlds.
- The current philosophy of abnormal worlds has them as ‘worlds’ that go beyond the logically possible.
- So, we want our alethic necessity operator to look only at normal worlds—only at worlds that are within logical limits (so to speak).
Proposition 2: UANW

The current idea is to make explicit use of our (sub-) collection $\mathcal{N} \subseteq \mathcal{W}$ of worlds, namely the normal worlds.

- We define our uniform, all-normal-worlds (UANW) Box thus:

  $$w \models \square A \text{ iff } w' \models A \text{ for all } w' \in \mathcal{N}$$

In short: for any world $w$ (of any sort), $\square A$ is true at $w$ iff $A$ is true at all normal worlds (versus, as in UAW, all worlds).

What do we get from the UANW approach?
VIRTUES: UANW

This approach has expected virtues, some of which are:
- Necessitation holds. (If $A$ is true at all normal worlds of all models, so is $\square A$ on the UANW approach!)
- Many (if not all) of the other minimal desiderata are satisfied.

...but not all robustly virtuous things are purely virtuous. The UANW has a severe, knockdown defect: it engenders Curry-driven triviality!!
Problem: UANW and PMP!!

Whatever its other virtues, the UANW approach is untenable. Recall the trouble with PMP and Curry paradoxes.

- Define: let $A \Rightarrow B$ be $\Box (A \rightarrow B)$.
- Claim: $\vdash (A \land (A \Rightarrow B)) \Rightarrow B$.
- Proof: suppose $w \not\models_1 \Box (A \land \Box (A \rightarrow B) \rightarrow B)$ for some $w \in \mathcal{N}$, in which case there’s some $x \in \mathcal{N}$ such that $x \not\models A \land \Box (A \rightarrow B) \rightarrow B$, and so there’s some $y \in \mathcal{W}$ such that $y \models A$ and $y \models \Box (A \rightarrow B)$ but $y \not\models B$. As $y \models \Box (A \rightarrow B)$ we have $z \models A \rightarrow B$ for all $z \in \mathcal{N}$, and so no world (including $y$) makes $A$ but not $B$ true. Contradiction.
So, we get PMP for $\Rightarrow$. And this delivers Curry-driven triviality all over again.

- NB: T-biconditionals using $\rightarrow$ imply the $\Rightarrow$ version.
- So, just run the original Curry problem for our arrow all over again, replacing $\rightarrow$ with $\Rightarrow$.
- NB: what this shows is that, in Greg Restall’s terminology, our theory is no longer robustly contraction-free: the UANW approach to $\Box$ creates a ‘contracting conditional’.

What to do??!!
Diagnosis: UANW and PMP

- Curry paradox taught that our regular arrow had to be *jumpy*; it had to behave differently at abnormal worlds than at normal ones.
- On our UANW approach, it doesn’t matter where in our universe of worlds we are (e.g., a normal or abnormal point); Box claims always look back to normal worlds.
- What’s going on, then, is that our UANW approach to $\Box A$ forces $A$ to be evaluated at normal points.
- And that’s the problem: PMP is broken only by evaluating parts of it at abnormal points; and $\Box$-ed PMP doesn’t get that choice.
PROPOSAL 3: JUMPY ALL-NORMAL-WORLDS

As with our conditional, so too with our Box: Curry paradox teaches that our Box is jumpy.

- For all normal worlds $w \in \mathcal{N}$:

  $$w \models \Box A \text{ iff } w' \models A \text{ for all } w' \in \mathcal{N}$$

- For all abnormal worlds $w \in \mathcal{W} \setminus \mathcal{N}$:

  $$w \models \Box A \text{ iff } w \models A$$

The ‘normal’ clause retains the UANW spirit, but is no longer uniform; it applies only to normal worlds. The ‘abnormal’ clause treats the Box as a truth operator at abnormal worlds.
GOOD NEWS!

- We avoid ⇒-PMP!! Countermodel: $\mathcal{W} = \{x, y\}$ with $y \in \mathcal{W} \setminus \mathcal{N}$. Let $y \models A$ and $y \models A \rightarrow B$ (say, on arbitrary evaluator or etc.) but $y \not\models B$. Then $y \models A \land \Box(A \rightarrow B)$ but, as above, $y \not\models B$.
- We get Necessitation: if $\vdash A$ then $\vdash \Box A$.
- K: $\Box(A \rightarrow B) \vdash \Box A \rightarrow \Box B$. (Not axiom!)
- S4: $\vdash \Box A \rightarrow \Box \Box A$
REMAINING ISSUES: S5 PRINCIPLES AND NEGATION

- Taking $\Diamond$ as $\neg \Box \neg$: the answer turns on negation.
  - Note: on the Star treatment, there’s mixed news:
    
    \[ \Diamond A \not\iff \Box \Diamond A \]
    
    but
    
    \[ \Diamond \Box A \vdash \Box A \]
    
    - On an alternative (non-star) approach to negation, one may get both versions (though perhaps not axiom).
- Taking $\Diamond$ as primitive will probably give both options even in Star setting (but I’d prefer not to take it as primitive).
I think that we get a relatively straightforward non-triviality proof for the resulting theory, but details remain open here.
CLOSING REMARKS

- Dialetheists posit worlds in order to enjoy a proper conditional.
- Curry paradox teaches that abnormal worlds are also required for the conditional, and that the conditional needs to be ‘jumpy’.
- In such a setting, it is natural to ask about broad necessity (or, dually, possibility): how does it work?
- I’ve shown that, at least in the target theories, Curry paradox reemerges when alethic necessity is added.
- I’ve also shown that the Curry-paradoxical lesson is the same: necessity, in the end, is abnormal—jumpy.
DISCUSSION IS OPEN!!!

...may discussion, unlike necessity, be uniform and normal.